

Canonical transformations for fermions in superanalysis

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Abstract

Canonical transformations (Bogoliubov transformations) for fermions with an infinite number of degrees of freedom are studied within a calculus of superanalysis. A continuous representation of the orthogonal group is constructed on a Grassmann module extension of the Fock space. The pull-back of these operators to the Fock space yields a unitary ray representation of the group that implements the Bogoliubov transformations.

1 Introduction

Canonical transformations for fermions have been introduced by Bogoliubov [9, 10] and by Valatin [28, 29] to diagonalize Hamiltonians of the theory of superconductivity. Canonical transformations for systems with an infinite number of degrees of freedom have been studied for bosons and for fermions in the book of Friedrichs [12]. Mathematically minded investigations for fermionic systems are often based on a study of the Clifford algebra of the field operators, cf. e.g. [27, 1, 2, 5, 4]. The group theoretical structure of the canonical transformations for fermions is that of the (infinite dimensional) orthogonal group, which acts on the real Hilbert space that underlies the complex one particle Hilbert space. An alternative approach to canonical transformations is therefore the construction of a unitary representation of this orthogonal group. After partial solutions e.g. in the books [12, 7] a complete construction has been given by Ruijsenaars [25, 26] with rigorous normal ordering expansions.

In this paper we present fermionic canonical transformations with the methods of infinite dimensional superanalysis as presented in Ref. [19]. This approach of superanalysis uses Grassmann modules with a Hilbert norm in contrast to the standard literature, which either concentrates on the algebraic structure of finite dimensional superanalysis and supermanifolds [8, 11], or – following [23] – uses a Banach norm for the superalgebra, cf. e. g. [15, 17, 24]. The aim of the paper is twofold. In the first part we recapitulate and amend the genuine infinite dimensional superanalysis of Ref. [19]. The main tool is the Grassmann module extension of the fermionic Fock space. In the second part of the paper a representation of the orthogonal group is constructed on the linear span of fermionic coherent vectors, which exist in the module Fock space. Then the pull-back of the module operators to the physical Fock space leads to a unitary ray representation of the orthogonal group.

The plan of the paper is as follows. In Sec. 2 first some facts about Hilbert and Fock spaces are recapitulated. Then an essentially self-contained presentation of superanalysis in infinite dimensional spaces follows. Superanalysis allows to define coherent vectors and Weyl operators also for fermions. Weyl operators and their interplay with canonical transformations on the fermionic Fock space are discussed in Sec. 3. The properties of the infinite dimensional

orthogonal group are reviewed in Sec. 4. The construction of the representation of the orthogonal group on the module Fock space and the pull-back to the physical Fock space is given in Sec. 5. The subclass of orthogonal transformations, for which the transformed vacuum has still an overlap with the old vacuum, is investigated in Sec. 5.1. The representation of these transformations is given with the methods of superanalysis on the linear span of coherent vectors. These calculations are the fermionic counterpart to the representation of the bosonic canonical transformations in Ref. [18]. The representation of the full orthogonal group follows in Sec. 5.2. The orbit of the vacuum generated by all canonical transformations is given in Sec. 5.3. Some proofs and detailed calculations are postponed to the Appendices A and B.

2 Fock spaces and superanalysis

2.1 The Fock space of antisymmetric tensors

In this Section we recapitulate some basic statements about Hilbert spaces and Fock spaces of antisymmetric tensors. Let \mathcal{H} be a complex separable Hilbert space with inner product $(f | g)$ and with an antiunitary involution $f \rightarrow f^*$, $f^{**} \equiv (f^*)^* = f$. Then $\langle f | g \rangle := (f^* | g) \in \mathbb{C}$ is a symmetric \mathbb{C} -bilinear form $\langle f | g \rangle = \langle g | f \rangle$, $f, g \in \mathcal{H}$. The underlying real Hilbert space of \mathcal{H} is denoted as $\mathcal{H}_{\mathbb{R}}$. This space has the inner product $(f | g)_{\mathbb{R}} = \text{Re } (f | g)$.

We use the following notations for linear operators. The space of all bounded operators A with operator norm $\|A\|$ is $\mathcal{L}(\mathcal{H})$. The adjoint operator is denoted by A^\dagger . The complex conjugate operator \bar{A} and the transposed operator A^T are defined by the identities $\bar{A}f = (Af^*)^*$ and $A^T f = (A^\dagger f^*)^*$ for all $f \in \mathcal{H}$. The usual relations $A^\dagger = (\bar{A})^T = \overline{(A^T)}$ are valid. An operator $A \in \mathcal{L}(\mathcal{H})$, which has the property $A^T = \pm A$, satisfies the symmetry relation $\langle f | Ag \rangle = \pm \langle Af | g \rangle$ for all $f, g \in \mathcal{H}$. It is called transposition-symmetric or skew symmetric, respectively. The space of all Hilbert-Schmidt (HS) operators A with norm $\|A\|_2 = \sqrt{\text{tr}_{\mathcal{H}} A^\dagger A}$ is $\mathcal{L}_2(\mathcal{H})$, the space of all trace class or nuclear operators A with norm $\|A\|_1 = \text{tr}_{\mathcal{H}} \sqrt{A^\dagger A}$ is $\mathcal{L}_1(\mathcal{H})$. The HS operators with the property $A^T = \pm A$ form a closed subspace within $\mathcal{L}_2(\mathcal{H})$ for which the notation $\mathcal{L}_2^\pm(\mathcal{H})$ is used. The space of all unitary operators in $\mathcal{L}(\mathcal{H})$ is called $\mathcal{U}(\mathcal{H})$. Projection operator always means an orthogonal projection.

The antisymmetric tensor product or exterior product is written with the symbol \wedge . The linear span of all tensors $f_1 \wedge \cdots \wedge f_n$, $f_j \in \mathcal{H}$, $j = 1, \dots, n$, is denoted as $\mathcal{H}^{\wedge n}$. The space $\mathcal{H}^{\wedge 0}$ is the one dimensional space \mathbb{C} . The Hilbert norm of the space $\mathcal{H}^{\wedge n}$ is written as $\|\cdot\|_n$ and the exterior product of the vectors $f_j \in \mathcal{H}$, $j = 1, \dots, n$, is normalized to $\|f_1 \wedge \cdots \wedge f_n\|_n^2 = \det(f_i | f_j)$. The completion of the space $\mathcal{H}^{\wedge n}$ with the norm $\|\cdot\|_n$ is the Hilbert space $\mathcal{A}_n(\mathcal{H})$. The exterior product extends by linearity to the linear space of tensors of finite degree $\mathcal{A}_{fin}(\mathcal{H}) = \cup_{N=0}^\infty \oplus_{n=0}^N \mathcal{A}_n(\mathcal{H})$. This space is an (infinite dimensional) Grassmann algebra. The unit is the normalized basis vector 1_{vac} (vacuum) of the space $\mathcal{H}^{\wedge 0} = \mathbb{C}$. If $\mathcal{F} \subset \mathcal{H}$ is a closed subspace of \mathcal{H} then $\mathcal{A}_n(\mathcal{F})$ is the completed linear span of all tensors $f_1 \wedge \cdots \wedge f_n$, $f_j \in \mathcal{F}$, and $\mathcal{A}_{fin}(\mathcal{F}) := \cup_{N=0}^\infty \oplus_{n=0}^N \mathcal{A}_n(\mathcal{F})$ is a subalgebra of $\mathcal{A}_{fin}(\mathcal{H})$.

Any element $F \in \mathcal{A}_{fin}(\mathcal{H})$ can be decomposed as $F = \sum_{n=0}^\infty F_n$, $F_n \in \mathcal{A}_n(\mathcal{H})$, and it is given the norm

$$\|F\|^2 = \sum_{n=0}^\infty \|F_n\|_n^2. \quad (1)$$

The completion of $\mathcal{A}_{fin}(\mathcal{H})$ with this norm is the standard Fock space of antisymmetric tensors $\mathcal{A}(\mathcal{H}) = \oplus_{n=0}^\infty \mathcal{A}_n(\mathcal{H})$. The inner product of two elements $F, G \in \mathcal{A}(\mathcal{H})$ is written as $(F | G)$.

The antiunitary involution $f \rightarrow f^*$ on \mathcal{H} can be extended uniquely to an antiunitary involution $F \rightarrow F^*$ on $\mathcal{A}(\mathcal{H})$ with the rule $(F \wedge G)^* = G^* \wedge F^*$. Then $\langle F \parallel G \rangle := (F^* \mid G)$ is a \mathbb{C} -bilinear symmetric form on $\mathcal{A}(\mathcal{H})$. Any bounded operator $B \in \mathcal{L}(\mathcal{H})$ can be lifted to an operator $\Gamma(B)$ on $\mathcal{A}(\mathcal{H})$ by the rules $\Gamma(B)1_{vac} = 1_{vac}$ and $\Gamma(B)(f_1 \wedge \dots \wedge f_n) := Bf_1 \wedge \dots \wedge Bf_n$, $f_j \in \mathcal{H}$, $n \in \mathbb{N}$. The operator $\Gamma(B)$ is a contraction, if B is a contraction, and it is isometric/unitary, if B is isometric/unitary.

For products of vectors we use the following notation. Let $\mathbf{A} \subset \mathbb{N}$ be a finite subset of the natural numbers with cardinality $|\mathbf{A}| = n \geq 1$. Then the elements of \mathbf{A} can be ordered $\mathbf{A} = \{a_1 < \dots < a_n\}$. Given the vectors f_a , $a \in \mathbf{A}$ the tensor $f_{\mathbf{A}}$ is defined as the product $f_{a_1} \wedge \dots \wedge f_{a_n}$ with ordered indices. For the empty set $\mathbf{A} = \emptyset$ we define $f_{\emptyset} = 1_{vac}$. If f_b , $b \in \mathbf{B}$, is another family of vectors, indexed by the finite set $\mathbf{B} \subset \mathbb{N}$, $\mathbf{A} \cap \mathbf{B} = \emptyset$, then the exterior product of these vectors is $f_{\mathbf{A}} \wedge f_{\mathbf{B}} = (-1)^{\tau(\mathbf{A}, \mathbf{B})} f_{\mathbf{A} \cup \mathbf{B}}$. The exponent $\tau(\mathbf{A}, \mathbf{B}) := \#\{(a, b) \in \mathbf{A} \times \mathbf{B} \mid a > b\}$ counts the number of inversions. In the same notation we write $z_{\mathbf{A}}$ for the product $z_{a_1} z_{a_2} \dots z_{a_n}$ of the complex numbers z_a , $a \in \mathbf{A}$, with $z_{\emptyset} = 1$. The symbol $\sum_{\mathbf{A} \subset \mathbb{N}}$ always means summation over the power set $\mathcal{P}(\mathbb{N})$ of \mathbb{N} , i.e. over all *finite* subsets of \mathbb{N} including the empty set.

2.2 Superanalysis and coherent vectors

Superanalysis allows to define coherent vectors and Weyl operators also for fermions. An approach of superanalysis for spaces with infinite dimensions has been developed in [19]. In this Section we present a slightly modified and extended version of superanalysis, which is used in the subsequent calculations. Some proofs are given in the Appendix A or in Ref. [19]. For the superalgebra Λ we choose a Grassmann algebra Λ which differs by notations and a weaker Hilbert norm from the Fock space $\mathcal{A}(\mathcal{H})$. The Grassmann algebra is the direct sum $\Lambda = \oplus_{p \geq 0} \Lambda_p$ of the subspaces Λ_p of tensors of degree p . The generating space is the infinite dimensional Hilbert space Λ_1 . The space Λ_p is given the Hilbert norm $\|\cdot\|_p$ of antisymmetric tensors of degree p normalized to the determinant (as for $\mathcal{A}_p(\mathcal{H})$). The unit is denoted by κ_0 , and it has the norm $\|\kappa_0\|_0 = 1$. The topology of Λ is then defined by the Hilbert norm

$$\|\lambda\|_{\Lambda}^2 = \sum_{p=0}^{\infty} (p!)^{-2} \|\lambda_p\|_p^2 \quad (2)$$

if $\lambda = \sum_{p=0}^{\infty} \lambda_p$, $\lambda_p \in \Lambda_p$. The antisymmetric tensor product of two elements λ_1 and λ_2 of Λ is now denoted as Grassmann product and it is written as $\lambda_1 \lambda_2$. As consequence of the topology (2) this product is continuous with the estimate $\|\lambda_1 \lambda_2\|_{\Lambda} \leq \sqrt{3} \|\lambda_1\|_{\Lambda} \|\lambda_2\|_{\Lambda}$, cf. [19] Appendix A or [20]. If $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda$ the stronger estimate $\|\lambda_1 \lambda_2\|_{\Lambda} \leq \|\lambda_1\|_{\Lambda} \|\lambda_2\|_{\Lambda}$ follows as for the standard Fock space. This Grassmann algebra is a superalgebra $\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}$ with the even part $\Lambda_{\bar{0}}$ and the odd part $\Lambda_{\bar{1}}$ consisting of tensors of even or odd degree, respectively. As additional structure we introduce an antiunitary involution $\kappa \rightarrow \kappa^*$ in the generating space Λ_1 . This involution is extended to an antiunitary involution $\lambda \rightarrow \lambda^*$ on Λ by the usual rules.

Remark 1 *The Grassmann algebra Λ is needed for a correct bookkeeping of the fermionic degrees of freedom. There is no canonical way for such constructions, and instead of a Grassmann algebra one can take a more general superalgebra Λ . In the literature about superanalysis the topology of Λ is either not discussed, or – following [23] – the Grassmann algebra is equipped with a Banach space topology, cf. e.g. [15, 17, 24]. But a Hilbert space topology is easier to handle, and the pull-back to the Fock space of standard physics is more transparent.*

The Hilbert space \mathcal{H} , the algebra of antisymmetric tensors $\mathcal{A}_{fin}(\mathcal{H})$ and the Fock space $\mathcal{A}(\mathcal{H})$ can be extended to the Λ -modules $\mathcal{H}^\Lambda = \Lambda \widehat{\otimes} \mathcal{H}$, $\mathcal{A}_{fin}^\Lambda(\mathcal{H}) = \bigcup_{N=0}^\infty \bigoplus_{n=0}^N \Lambda \widehat{\otimes} \mathcal{A}_n(\mathcal{H})$ and $\mathcal{A}^\Lambda(\mathcal{H}) = \Lambda \widehat{\otimes} \mathcal{A}(\mathcal{H})$.¹ The Hilbert norm of the module Fock space $\mathcal{A}^\Lambda(\mathcal{H})$ is denoted as $\|\cdot\|_\otimes$. The Λ -linear space $\mathcal{A}_{fin}^\Lambda(\mathcal{H})$ is a pre-Hilbert space with completion $\mathcal{A}^\Lambda(\mathcal{H})$. The tensor product of $\mathcal{A}_{fin}(\mathcal{H})$ has a Λ -linear extension $\mathcal{A}_{fin}^\Lambda(\mathcal{H}) \ni \Xi_1, \Xi_2 \rightarrow \Xi_1 \circ \Xi_2 \in \mathcal{A}_{fin}^\Lambda(\mathcal{H})$. This product is uniquely determined by the rule $\Xi_1 \circ \Xi_2 = \lambda_1 \lambda_2 \otimes (F_1 \wedge F_2)$ if $\Xi_j = \lambda_j \otimes F_j \in \Lambda \otimes \mathcal{A}_{fin}(\mathcal{H})$, $j = 1, 2$. Any tensor Ξ can be written as series $\Xi = \sum_{p=0, n=0}^\infty \Xi_{p,n}$ with $\Xi_{p,n} \in \Lambda_p \widehat{\otimes} \mathcal{A}_n(\mathcal{H})$. If the set $\{p+n \mid \Xi_{p,n} \neq 0\}$ contains only even (odd) numbers, we say the tensor Ξ has even (odd) parity, $\pi(\Xi) = 0(1)$. With this parity the module $\mathcal{A}_{fin}^\Lambda(\mathcal{H})$ is a \mathbb{Z}_2 -graded algebra. If the tensors Θ and Ξ have the parities $\pi(\Theta) = p$ and $\pi(\Xi) = q$, the product satisfies $\Theta \circ \Xi = (-1)^{pq} \Xi \circ \Theta$. The product $\Theta \circ \Xi$ is not defined on the full the module Fock space $\mathcal{A}^\Lambda(\mathcal{H})$, but it can be extended to a larger class than $\mathcal{A}_{fin}^\Lambda(\mathcal{H})$ by continuity arguments, cf. Appendix A.1 and Lemma 2.

The inner product of $\mathcal{A}(\mathcal{H})$ has a unique module extension to $\mathcal{A}^\Lambda(\mathcal{H})$ with the property $(\lambda \otimes F \mid \mu \otimes G) = \lambda^* \mu (F \mid G) \in \Lambda$ where $\lambda, \mu \in \Lambda$ and $F, G \in \mathcal{A}(\mathcal{H})$. This Λ -valued inner product $\mathcal{A}^\Lambda(\mathcal{H}) \times \mathcal{A}^\Lambda(\mathcal{H}) \ni (\Theta, \Xi) \rightarrow (\Theta \mid \Xi) \in \Lambda$ has the hermiticity property $(\Theta \mid \Xi)^* = (\Xi \mid \Theta)$, and it satisfies the norm estimate $\|(\Theta \mid \Xi)\|_\Lambda \leq \sqrt{3} \|\Theta\|_\otimes \|\Xi\|_\otimes$, cf. [19]. If the tensors Θ and Ξ have the same parity, $\pi(\Theta) = \pi(\Xi)$, the form $(\Theta \mid \Xi) \in \Lambda_{\bar{0}}$ is an even element of Λ . The involution of $\mathcal{A}(\mathcal{H})$ can be extended to an involution $\Xi \rightarrow \Xi^*$ on $\mathcal{A}^\Lambda(\mathcal{H})$ with the property $(\lambda \otimes F)^* = \lambda^* \otimes F^*$, and the bilinear form $\langle \cdot \mid \cdot \rangle$ of $\mathcal{A}(\mathcal{H})$ has a unique Λ -extension $\mathcal{A}^\Lambda(\mathcal{H}) \times \mathcal{A}^\Lambda(\mathcal{H}) \ni (\Theta, \Xi) \rightarrow \langle \Theta \mid \Xi \rangle := (\Theta^* \mid \Xi) \in \Lambda$.

The factorization property of the inner product of $\mathcal{A}(\mathcal{H})$ for tensors in orthogonal subspaces leads to the following factorization of $(\Theta \mid \Xi)$:

Lemma 1 *Let \mathcal{H}_1 and \mathcal{H}_2 be two orthogonal closed subspaces of the Hilbert space \mathcal{H} . The tensors $\Theta_j, \Xi_j \in \mathcal{A}^\Lambda(\mathcal{H})$, $j = 1, 2$, are restricted by :*

(i) *The tensors are elements of the orthogonal subspaces, $\Theta_1, \Xi_1 \in \mathcal{A}^\Lambda(\mathcal{H}_1)$ and*

$\Theta_2, \Xi_2 \in \mathcal{A}^\Lambda(\mathcal{H}_2)$.

(ii) *The tensors Θ_1 and Ξ_1 have equal parity.*

Then the products $\Theta_1 \circ \Theta_2$ and $\Xi_1 \circ \Xi_2$ are defined and the factorization $(\Theta_1 \circ \Theta_2 \mid \Xi_1 \circ \Xi_2) = (\Theta_1 \mid \Xi_1) (\Theta_2 \mid \Xi_2)$ is true.

If $\mu \in \Lambda$ and $T \in \mathcal{L}(\mathcal{A}(\mathcal{H}))$ then there is a unique bounded operator $\mu \otimes T$ on $\mathcal{A}^\Lambda(\mathcal{H})$ that maps $\lambda \otimes F \in \Lambda \otimes \mathcal{A}(\mathcal{H})$ onto $\mu \lambda \otimes T F \in \Lambda \otimes \mathcal{A}(\mathcal{H})$.

Definition 1 *We call a continuous \mathbb{C} -linear operator \hat{T} on $\mathcal{A}^\Lambda(\mathcal{H}) = \Lambda \widehat{\otimes} \mathcal{A}(\mathcal{H})$ a regular operator, if it can be represented as $\hat{T} = \sum_{j \in \mathbf{J}} \mu_j \otimes T_j$ with $\mu_j \in \Lambda$, $T_j \in \mathcal{L}(\mathcal{A}(\mathcal{H}))$ and a finite or countable index set $\mathbf{J} \subset \mathbb{N}$. The series $\sum_{j \in \mathbf{J}} \|\mu_j\|_\Lambda \|T_j\|$ has to converge.*

The operator norm of $\hat{T} = \sum_{j \in \mathbf{J}} \mu_j \otimes T_j$ has the upper bound $\|\hat{T}\| \leq \sqrt{3} \sum_{j \in \mathbf{J}} \|\mu_j\|_\Lambda \|T_j\|$. If \hat{T}_1 and \hat{T}_2 are regular operators, then the product $\hat{T}_1 \hat{T}_2$ is a regular operator. Let \hat{T} be a bounded operator on $\mathcal{A}^\Lambda(\mathcal{H})$, then the operator \hat{T}^+ is called the superadjoint operator of \hat{T} , if the identity $(\Theta \mid \hat{T} \Xi) = (\hat{T}^+ \Theta \mid \Xi)$ is valid for all $\Theta, \Xi \in \mathcal{A}^\Lambda(\mathcal{H})$. For a regular operator $\hat{T} = \sum_j \lambda_j \otimes T_j$ the superadjoint operator is the regular operator $\hat{T}^+ = \sum_j \lambda_j^* \otimes T_j^\dagger$. In

¹The tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of two Hilbert spaces means the algebraic tensor space. The completion of $\mathcal{H}_1 \otimes \mathcal{H}_2$ with the Hilbert cross norm is $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$.

general the superadjoint operator is not the adjoint operator in the standard definition using the \mathbb{C} -valued inner product of the Hilbert space $\mathcal{A}^\Lambda(\mathcal{H})$.

For the subsequent constructions we define the fermionic *superspace* as the completed tensor space $\mathcal{H}_\Lambda := \Lambda_1 \widehat{\otimes} \mathcal{H}$. The algebraic superspace $\mathcal{H}_\Lambda^{alg} := \Lambda_1 \otimes \mathcal{H}$ is dense in \mathcal{H}_Λ . The inclusions $\mathcal{H}_\Lambda^{alg} \subset \mathcal{H}_\Lambda \subset \mathcal{H}^\Lambda \subset \mathcal{A}^\Lambda(\mathcal{H})$ are obvious. The elements of \mathcal{H}_Λ have even parity, and for $\xi, \eta \in \mathcal{H}_\Lambda$ the Λ -extended inner product has the hermiticity properties $(\xi | \eta)^* = (\eta | \xi) = -(\xi^* | \eta^*)$.

Remark 2 In Ref. [19] the fermionic part of the superspace is introduced as $\Lambda_{\bar{1}} \widehat{\otimes} \mathcal{H}$ with the full odd subspace $\Lambda_{\bar{1}}$ of the Grassmann algebra Λ , and the coherent vectors $\exp \xi$ are defined with arguments $\xi \in \Lambda_{\bar{1}} \widehat{\otimes} \mathcal{H}$. Here we use the strictly smaller superspace $\mathcal{H}_\Lambda = \Lambda_1 \widehat{\otimes} \mathcal{H}$ with the generating Hilbert space Λ_1 of Λ . In [19] this space was called *restricted superspace*. Calculations with $\Lambda_1 \widehat{\otimes} \mathcal{H}$ allow better norm estimates.

In choosing the superspace $\mathcal{H}_\Lambda = \Lambda_1 \widehat{\otimes} \mathcal{H}$ the multiplication with supervectors $\xi \in \mathcal{H}_\Lambda$ has the following property:

Lemma 2 Let $\xi \in \mathcal{H}_\Lambda = \Lambda_1 \widehat{\otimes} \mathcal{H}$ be a supervector, then the mapping $\mathcal{A}_{fin}^\Lambda(\mathcal{H}) \ni \Theta \rightarrow \xi \circ \Theta \in \mathcal{A}_{fin}^\Lambda(\mathcal{H})$ can be extended to a continuous mapping on $\mathcal{A}^\Lambda(\mathcal{H})$ with the norm estimate

$$\|\xi \circ \Theta\|_\otimes \leq \|\xi\|_\otimes \|\Theta\|_\otimes, \quad \Theta \in \mathcal{A}^\Lambda(\mathcal{H}). \quad (3)$$

The proof of this Lemma is given in the Appendix A.1.

If A is a bounded operator on \mathcal{H} , then $\kappa_0 \otimes A$ is a bounded operator on \mathcal{H}_Λ that maps $\xi = \mu \otimes f \in \mathcal{H}_\Lambda$ onto $(\kappa_0 \otimes A)\xi = \mu \otimes Af \in \mathcal{H}_\Lambda$. To simplify the notations we write $A\xi$ instead of $(\kappa_0 \otimes A)\xi$.

The fermionic coherent vectors are defined as [19]

$$\mathcal{H}_\Lambda \ni \xi \rightarrow \exp \xi = \sum_{p=0}^{\infty} (p!)^{-1} \xi^p \in \mathcal{A}^\Lambda(\mathcal{H}). \quad (4)$$

The norm convergence of this series follows from the estimate (3) with $\|\exp \xi\|_\otimes^2 \leq \sum_p (p!)^{-2} \|\xi^p\|_\otimes^2 \leq \sum_p (p!)^{-2} \|\xi\|_\otimes^{2p}$. The exponential $\exp \xi$ is therefore an entire analytic function on \mathcal{H}_Λ , and the usual multiplication rule $(\exp \xi) \circ (\exp \eta) = \exp(\xi + \eta)$ holds. The exponential vectors have even parity. If $\{e_k \mid k \in \mathbb{N}\}$ is an ON basis of \mathcal{H} , the tensors $\{e_{\mathbf{K}} \mid \mathbf{K} \subset \mathbb{N}\}$ are an ON basis of $\mathcal{A}(\mathcal{H})$. Any supervector has the representation $\xi = \sum_{k \in \mathbb{N}} \lambda_k \otimes e_k$ with elements $\lambda_k \in \Lambda_1$, $\sum_{k \in \mathbb{N}} \|\lambda_k\|_\Lambda^2 = \|\xi\|_\otimes^2$. Then the exponential vector is the series $\exp \xi = \sum_{\mathbf{K} \subset \mathbb{N}} \lambda_{\mathbf{K}} \otimes e_{\mathbf{K}}$, and $(\exp \xi | \exp \xi) = \sum_{\mathbf{K} \subset \mathbb{N}} (\lambda_{\mathbf{K}})^* \lambda_{\mathbf{K}} = \exp(\xi | \xi)$ follows. The Λ -inner product of two exponential vectors is therefore

$$(\exp \xi | \exp \eta) = \exp(\xi | \eta). \quad (5)$$

If Θ and Ξ are tensors in $\mathcal{A}^\Lambda(\mathcal{H})$, which have even parity, and for which the product $\Theta \circ \Xi$ is defined, the formula

$$(\exp \xi | \Theta \circ \Xi) = (\exp \xi | \Theta) (\exp \xi | \Xi) \quad (6)$$

is valid. This rule is obviously true for $\Theta = \exp \eta$ and $\Xi = \exp \zeta$ with $\eta, \zeta \in \mathcal{H}_\Lambda$. If η is a tensor of degree $2p$ and ζ is a tensor of degree $2q$, the identity (6) follows for the terms

with the highest tensor degrees, i.e. $\Theta = \eta^p$ and $\Xi = \zeta^q$. Some algebra – using techniques of Appendix B in [19] – leads to the rule for $\Theta, \Xi \in \mathcal{A}_{fin}^\Lambda(\mathcal{H})$ with even parity. Then continuity arguments extend the identity to those elements of $\mathcal{A}^\Lambda(\mathcal{H})$, for which $\Theta \circ \Xi$ is defined.

The \mathbb{C} -linear span of $\{\exp \xi \mid \xi \in \mathcal{H}_\Lambda\}$ is called $\mathcal{C}(\mathcal{H}_\Lambda)$, the Λ -linear span $\Lambda \otimes \mathcal{C}(\mathcal{H}_\Lambda)$ is called $\mathcal{C}^\Lambda(\mathcal{H}_\Lambda)$. With the identification $\mathcal{C}(\mathcal{H}_\Lambda) \simeq \kappa_0 \otimes \mathcal{C}(\mathcal{H}_\Lambda)$ the space $\mathcal{C}(\mathcal{H}_\Lambda)$ is a subspace of $\mathcal{C}^\Lambda(\mathcal{H}_\Lambda)$, and we have the inclusions $\mathcal{H}_\Lambda \subset \mathcal{C}(\mathcal{H}_\Lambda) \subset \mathcal{C}^\Lambda(\mathcal{H}_\Lambda) \subset \mathcal{A}^\Lambda(\mathcal{H})$. Despite the values taken by functions $\Phi \in \mathcal{C}(\mathcal{H}_\Lambda)$ only trivially intersect with the fermionic Fock space $\mathcal{A}(\mathcal{H})$ the following Lemmata are true.

Lemma 3 *A tensor $\Xi \in \mathcal{A}^\Lambda(\mathcal{H})$ is uniquely determined by the function $\mathcal{H}_\Lambda \ni \zeta \rightarrow (\exp \zeta \mid \Xi) \in \Lambda$.*

The proof of this Lemma follows from Sec. 5.3 of [19] Lemma 4 and Corollary 2.

Lemma 4 *Let \hat{T} be a regular operator on $\mathcal{A}^\Lambda(\mathcal{H})$. Then \hat{T} is uniquely determined by its values on the set of coherent vectors $\{\exp \xi \mid \xi \in \mathcal{H}_\Lambda\}$.*

Proof. The operator \hat{T} is a linear and continuous operator on the Hilbert space $\mathcal{A}^\Lambda(\mathcal{H})$. It is sufficient to prove that $\hat{T} \exp \xi = 0$, $\xi \in \mathcal{H}_\Lambda$, implies $\hat{T} \Xi = 0$ for $\Xi \in \mathcal{A}^\Lambda(\mathcal{H})$. Given the regular operator $\hat{T} = \sum_j \mu_j \otimes T_j$ it has the bounded superadjoint $\hat{T}^+ = \sum_j \mu_j^* \otimes T_j^\dagger$. Assume $\hat{T} \exp \xi = 0$ is true for all $\xi \in \mathcal{H}_\Lambda$. Then $0 = (\hat{T} \exp \xi \mid \exp \eta) = (\exp \xi \mid \hat{T}^+ \exp \eta)$ holds for all $\xi, \eta \in \mathcal{H}_\Lambda$, and – as a consequence of Lemma 3 – the identity $\hat{T}^+ \exp \eta = 0$ is true for all $\eta \in \mathcal{H}_\Lambda$. But then we have $(\exp \xi \mid \hat{T} \Xi) = (\hat{T}^+ \exp \xi \mid \Xi) = 0$ for $\xi \in \mathcal{H}_\Lambda$ and $\Xi \in \mathcal{A}^\Lambda(\mathcal{H})$. Hence $\hat{T} \Xi = 0$ is true for $\Xi \in \mathcal{A}^\Lambda(\mathcal{H})$. \square

This Lemma implies

Corollary 1 *Let \hat{T} be a regular operator on $\mathcal{A}^\Lambda(\mathcal{H})$. Then \hat{T} is uniquely determined by the function $\mathcal{H}_\Lambda \ni \xi \rightarrow (\exp \xi \mid \hat{T} \exp \xi) \in \Lambda$.*

Proof. The Lemmata 3 and 4 imply that \hat{T} is determined by the function $\varphi(\xi, \eta) := (\exp \xi \mid \hat{T} \exp \eta)$, which is analytic in $\eta \in \mathcal{H}_\Lambda$ and antianalytic in $\xi \in \mathcal{H}_\Lambda$. Such a function is uniquely determined by its values on the diagonal $\xi = \eta$. \square

The Fock space creation and annihilation operators $a^+(h)F = h \wedge F$, $h \in \mathcal{H}$, $F \in \mathcal{A}(\mathcal{H})$, and $a^-(h) = (a^+(h))^\dagger$ have an extension to \mathbb{C} -linear operators $b^+(\eta)$ and $b^-(\eta)$, $\eta \in \mathcal{H}_\Lambda$, on $\mathcal{A}^\Lambda(\mathcal{H})$. The operator $b^+(\eta)$ is simply given by $b^+(\eta) \Xi = \eta \circ \Xi$ with $\eta \in \mathcal{H}_\Lambda$ for all $\Xi \in \mathcal{A}^\Lambda(\mathcal{H})$. The operator $b^-(\eta)$ is the superadjoint $b^-(\eta) = (b^+(\eta))^+$. For $\eta = \sum_j \mu_j \otimes f_j \in \mathcal{H}_\Lambda$ we obtain $b^+(\eta) = \sum_j \mu_j \otimes a^+(f_j)$ and $b^-(\eta) = \sum_j \mu_j^* \otimes a^-(f_j)$. The operators $b^\pm(\eta)$ are continuous with operator norm $\|b^\pm(\eta)\| \leq \|\eta\|_\otimes$, $\eta \in \mathcal{H}_\Lambda$, cf. the end of Appendix A.1. The mapping $\mathcal{H}_\Lambda \ni \eta \rightarrow b^+(\eta)$ is \mathbb{C} -linear, and $\eta \rightarrow b^-(\eta)$ is \mathbb{C} -antilinear. The operators $b^\pm(\eta)$ are regular operators. Lemma 4 implies that the operators $b^\pm(\eta)$ are uniquely determined by their values on coherent vectors $\exp \xi$, $\xi \in \mathcal{H}_\Lambda$. A simple calculation gives for all $\xi, \eta \in \mathcal{H}_\Lambda$

$$b^+(\eta) \exp \xi = \eta \cdot \exp \xi, \quad b^-(\eta) \exp \xi = (\eta \mid \xi) \exp \xi. \quad (7)$$

2.3 Exponentials of tensors of second degree

Given a skew symmetric HS operator X on \mathcal{H} , then there exists exactly one tensor $\Omega(X) \in \mathcal{A}_2(\mathcal{H})$ such that $\langle \Omega(X) \parallel f \wedge g \rangle = \langle f \parallel Xg \rangle$ for all $f, g \in \mathcal{H}$, cf. Appendix A.2. The exponentials of tensors in $\mathcal{A}_2(\mathcal{H})$ have been investigated in the literature, e.g. in Chap. 12 of [22]. But for completeness and to fix the normalizations we derive the following statements in Appendix A.2.

1. For $X \in \mathcal{L}_2^-(\mathcal{H})$ the exponential series

$$\exp \Omega(X) = 1_{vac} + \frac{1}{1!} \Omega(X) + \frac{1}{2!} \Omega(X) \wedge \Omega(X) + \dots \quad (8)$$

converges uniformly in $\mathcal{A}(\mathcal{H})$, and the mapping $\mathcal{L}_2^-(\mathcal{H}) \ni X \rightarrow \exp \Omega(X) \in \mathcal{A}(\mathcal{H})$ is entire analytic.

2. The inner product of two of such tensors is

$$(\exp \Omega(X) \mid \exp \Omega(Y)) = \sqrt{\det(I + X^\dagger Y)} = \sqrt{\det(I + Y X^\dagger)}, \quad X, Y \in \mathcal{L}_2^-(\mathcal{H}). \quad (9)$$

The mapping $\mathcal{A}(\mathcal{H}) \ni F \rightarrow \kappa_0 \otimes F \in \mathcal{A}^\Lambda(\mathcal{H})$ gives a natural embedding of the Fock space $\mathcal{A}(\mathcal{H})$ into the Λ -module $\mathcal{A}^\Lambda(\mathcal{H}) = \Lambda \widehat{\otimes} \mathcal{A}(\mathcal{H})$. The tensors (8) can therefore be taken as the elements $\Psi(X) := \kappa_0 \otimes \exp \Omega(X) = \exp(\kappa_0 \otimes \Omega(X))$ of $\mathcal{A}^\Lambda(\mathcal{H})$. In Appendix A.3 we prove that the products

$$\Psi(X, \xi) := (\exp \xi) \circ \Psi(X) = \Psi(X) \circ (\exp \xi) = \exp(\xi + \kappa_0 \otimes \Omega(X)) \quad (10)$$

of the coherent vectors (4) and of the exponentials $\Psi(X)$ are well defined elements of $\mathcal{A}^\Lambda(\mathcal{H})$ for all $X \in \mathcal{L}_2^-(\mathcal{H})$ and $\xi \in \mathcal{H}_\Lambda$. The mapping $\mathcal{L}_2^-(\mathcal{H}) \times \mathcal{H}_\Lambda \ni (X, \xi) \rightarrow \Psi(X, \xi) \in \mathcal{A}^\Lambda(\mathcal{H})$ is entire analytic. The vectors $\Psi(X, \xi)$ are called *ultracoherent vectors*, cf. the bosonic case in Ref. [18]. The Λ -inner product of a coherent vector with $\Psi(X, \eta)$ is, cf. Appendix A.3,

$$(\exp \xi \mid \Psi(X, \eta)) = \exp \left(\xi \mid \eta + \frac{1}{2} X \xi^* \right), \quad (11)$$

and the Λ -inner product of an ultracoherent vector with itself is calculated in Appendix A.3 as

$$(\Psi(X, \xi) \mid \Psi(X, \xi)) = \sqrt{\det(I + X^\dagger X)} \exp \left(\frac{1}{2} \langle \xi^* \parallel A \xi^* \rangle + \langle \xi^* \parallel B \xi \rangle + \frac{1}{2} \langle \xi \parallel A^\dagger \xi \rangle \right) \quad (12)$$

with the operators $A = X(I + X^\dagger X)^{-1}$ and $B = (I + X X^\dagger)^{-1}$. The mapping $\mathcal{H}_\Lambda \times \mathcal{H}_\Lambda \ni (\xi, \eta) \rightarrow \langle \xi \parallel T \eta \rangle = (\xi^* \mid T \eta) \in \Lambda_2$ is well defined for any bounded operator $T \in \mathcal{L}(\mathcal{H})$ with the norm estimate $\|\langle \xi \parallel T \eta \rangle\|_\Lambda \leq \|T\| \|\xi\|_\otimes \|\eta\|_\otimes$. If T is a skew symmetric operator, the form $\langle \xi \parallel T \eta \rangle$ is symmetric in ξ and η . The exponential series of these Λ_2 -valued forms is absolutely converging within Λ , cf. Appendix A.2.

3 Weyl operators and canonical transformations

3.1 Weyl operators for fermions

It is convenient to define Weyl operators for fermions $W(\eta)$ with $\eta \in \mathcal{H}_\Lambda$ first on $\mathcal{C}(\mathcal{H}_\Lambda)$ by their action on exponential vectors, as it has been done for bosons, cf. Sec. 3.1. of [18],

$$W(\eta) \exp \xi := e^{-(\eta|\xi) - \frac{1}{2}(\eta|\eta)} \exp(\eta + \xi) \in \Lambda_{\bar{0}} \otimes \mathcal{C}(\mathcal{H}_\Lambda). \quad (13)$$

The operators $W(\eta)$ form a group with $W(0) = id$, and

$$W(\xi)W(\eta) = e^{-i\omega(\xi, \eta)} W(\xi + \eta), \quad \xi, \eta \in \mathcal{H}_\Lambda. \quad (14)$$

Thereby $\omega(\xi, \eta)$ is the \mathbb{R} -bilinear antisymmetric form

$$\mathcal{H}_\Lambda \times \mathcal{H}_\Lambda \ni (\xi, \eta) \rightarrow \omega(\xi, \eta) = \frac{1}{2i} ((\xi | \eta) - (\eta | \xi)) \in \Lambda_2 \subset \Lambda_{\bar{0}}. \quad (15)$$

The relation (14) implies that $W(\eta)$ is invertible with $W^{-1}(\eta) = W(-\eta)$. The expectation of the Weyl operator between exponential vectors is calculated as

$$(\exp \xi | W(\eta) \exp \xi) = \exp((\xi | \xi) + (\xi | \eta) - (\eta | \xi) - \frac{1}{2}(\eta | \eta)) = (W(-\eta) \exp \xi | \exp \xi).$$

These identities imply the relation $W^+(\eta) = W(-\eta)$, valid on $\mathcal{C}(\mathcal{H}_\Lambda)$.

For fixed $\eta \in \mathcal{H}_\Lambda$ the operators $\mathbb{R} \ni t \rightarrow W(t\eta)$ form a one parameter group. The generator D_η of this group follows from (13) as $D_\eta \exp \xi = \frac{d}{dt} W(t\eta) \exp \xi|_{t=0} = -(\eta | \xi) \exp \xi + \eta \cdot \exp \xi$, i.e. $D_\eta = -b^-(\eta) + b^+(\eta)$, where $b^\pm(\eta)$ are the creation and annihilation operators (7). Since these operators are bounded operators on $\mathcal{A}^\Lambda(\mathcal{H})$, the Weyl operator can be defined by the exponential series expansion

$$W(\eta) = \exp(b^+(\eta) - b^-(\eta)) = e^{-\frac{1}{2}(\eta|\eta)} \exp(b^+(\eta)) \exp(-b^-(\eta)) \quad (16)$$

for $\eta \in \mathcal{H}_\Lambda$ as bounded operators on the module Fock space $\mathcal{A}^\Lambda(\mathcal{H})$. The operator (16) is a regular operator in the sense of Definition 1, and it agrees on $\mathcal{C}(\mathcal{H}_\Lambda)$ with (13). As a consequence of Lemma 4 the relations $W^{-1}(\eta) = W(-\eta) = W^+(\eta)$ are true on $\mathcal{A}^\Lambda(\mathcal{H})$, and the following identity is valid for $\eta \in \mathcal{H}_\Lambda$ and all tensors $\Xi, \Upsilon \in \mathcal{A}^\Lambda(\mathcal{H})$

$$(W(\eta) \Xi | W(\eta) \Upsilon) = (W^+(\eta) W(\eta) \Xi | \Upsilon) = (\Xi | \Upsilon). \quad (17)$$

In Appendix A.3 we calculate the action of $W(\eta)$ on the ultracoherent vectors (10) and obtain the formula

$$W(\eta) \Psi(X, \xi) = e^{-\frac{1}{2}(\eta|\eta) + \frac{1}{2}(\eta|X\eta^* - 2\xi)} \Psi(X, \xi + \eta - X\eta^*) \quad (18)$$

with $X \in \mathcal{L}_2^-$ and $\xi, \eta \in \mathcal{H}_\Lambda$. Finally, we list two properties of Weyl operators that are used in the subsequent Sections:

1. Let P be the projection operator onto the closed subspace $\mathcal{F} \subset \mathcal{H}$. If the restriction $PS = SP$ of the operator $S \in \mathcal{L}(\mathcal{H})$ is a unitary operator on \mathcal{F} , and if η is an element of $\Lambda \hat{\otimes} \mathcal{F} \subset \mathcal{H}_\Lambda$, then the following identity is true

$$\hat{\Gamma}(S)W(\eta)\hat{\Gamma}(S^\dagger P) = W(S\eta)\hat{\Gamma}(P). \quad (19)$$

2. Assume P_j , $j = 1, 2$, are projection operators onto orthogonal subspaces $\mathcal{H}_j = P_j \mathcal{H}$ of the Hilbert space \mathcal{H} , and $P = P_1 + P_2$ denotes the projection operator onto $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then the Weyl relation (14) implies the factorization into $W(P\eta) = W(P_1\eta)W(P_2\eta) = W(P_2\eta)W(P_1\eta)$. Under the additional assumption that $\Xi \in \mathcal{A}^\Lambda(\mathcal{H})$ is the product $\Xi = \Xi_1 \circ \Xi_2$ of the tensors $\Xi_j \in \mathcal{A}^\Lambda(\mathcal{H}_j)$, $j = 1, 2$, where Ξ_1 has the parity $\pi(\Xi_1) = k \in \{0, 1\}$ we obtain

$$W(P\eta)\Xi = (W(P_1\eta)\Xi_1) \circ \left(W((-1)^k P_2\eta)\Xi_2 \right). \quad (20)$$

The identity (19) is a consequence of the transformation behaviour $\Gamma(S)a^\pm(h)\Gamma(S^\dagger P) = a^\pm(Sh)\Gamma(P)$, $h \in \mathcal{F}$, of the creation/annihilation operators in $\mathcal{A}(\mathcal{H})$ under unitary transformations of \mathcal{H} . The identity (20) follows from (16) and the relations $b^\pm(P_1\eta)\Xi = (b^\pm(P_1\eta)\Xi_1) \circ \Xi_2$, $b^\pm(P_2\eta)\Xi = (-1)^k \Xi_1 \circ (b^\pm(P_2\eta)\Xi_2)$ for the module creation/annihilation operators (7).

3.2 Canonical transformations

The canonical anticommutation relations (CAR) for the Fock space creation and annihilation operators $\{a^-(f), a^+(g)\} \equiv a^-(f)a^+(g) + a^+(g)a^-(f) = (f | g)I$ and $\{a^+(f), a^+(f)\} = \{a^-(f), a^-(f)\} = 0$ can be written in the more condensed form using the antihermitean difference $\Delta(f) := a^+(f) - a^-(f)$. This difference is a function on $\mathcal{H}_\mathbb{R}$ – the underlying real space of \mathcal{H} – which has the inner product $\mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R} \ni (f, g) \rightarrow (f | g)_\mathbb{R} = \text{Re } (f | g) \in \mathbb{R}$. The anticommutation relations

$$\Delta(f)\Delta(g) + \Delta(g)\Delta(f) = -2(f | g)_\mathbb{R}I \text{ with } f, g \in \mathcal{H}_\mathbb{R} \quad (21)$$

are equivalent to the CAR of the creation and annihilation operators. In Sec. 5 we construct unitary operators T on $\mathcal{A}(\mathcal{H})$ that implement \mathbb{R} -linear transformations of the creation and annihilation operators

$$Ta^\pm(f)T^\dagger = a^\pm(Uf) - a^\mp(Vf^*) \quad (22)$$

such that the canonical anticommutation relations remain unchanged. Here U and V are bounded linear transformations on \mathcal{H} . The transformations $a^\pm(f) \rightarrow a^\pm(Uf) - a^\mp(Vf^*)$ are called canonical transformations or *Bogoliubov transformations*. Using the operators $\Delta(f)$, $f \in \mathcal{H}_\mathbb{R}$, the transformation rules (22) can be combined into

$$T\Delta(f)T^\dagger = \Delta(Uf + Vf^*). \quad (23)$$

The mapping

$$f \in \mathcal{H}_\mathbb{R} \rightarrow R(U, V)f := Uf + Vf^* \in \mathcal{H}_\mathbb{R}, \quad (24)$$

with $U, V \in \mathcal{L}(\mathcal{H})$ is the general continuous \mathbb{R} -linear mapping on $\mathcal{H}_\mathbb{R}$. The canonical anticommutation relations (21) are preserved, if the inner product of $\mathcal{H}_\mathbb{R}$ is invariant against this mapping, i.e. if

$$(R(U, V)f | R(U, V)g)_\mathbb{R} = (Uf + Vf^* | Ug + Vg^*)_\mathbb{R} = (f | g)_\mathbb{R} \quad (25)$$

holds for all $f, g \in \mathcal{H}_\mathbb{R}$. An invertible operator $R(U, V)$ which satisfies this identity is an orthogonal transformation of the space $\mathcal{H}_\mathbb{R}$. Such transformations actually form a group, which is discussed in more detail in Sec. 4. For infinite dimensional Hilbert spaces \mathcal{H} –

needed for quantum field theory – an additional constraint turns out to be necessary: In order to obtain a unitary representation, which has a vacuum state, the operator V has to be a HS operator [12, 27, 4]. This restricted orthogonal group of the space $\mathcal{H}_{\mathbb{R}}$ will be called $\mathcal{O}_2(\mathcal{H}_{\mathbb{R}})$.

There exists a large number of publications, in which unitary representations of this group on $\mathcal{A}(\mathcal{H})$ are investigated, cf. e.g. [12, 7, 26, 21]. It is the aim of this paper to construct a ray representation of the group $\mathcal{O}_2(\mathcal{H}_{\mathbb{R}})$ with the methods of superanalysis as presented in Sects. 2 and 3. That amounts to the construction of a representation by continuous operators $\hat{T}(R)$, $R \in \mathcal{O}_2(\mathcal{H}_{\mathbb{R}})$, acting on the module Fock space $\mathcal{A}^{\Lambda}(\mathcal{H})$ with the following property: $\hat{T}(R)$ is the product $\hat{T}(R) = \kappa_0 \otimes T(R)$ where $T(R)$ is a unitary operator on $\mathcal{A}(\mathcal{H})$. Using the creation and annihilation operators (7) and $D_{\xi} = b^{+}(\xi) - b(\xi)$, $\xi \in \mathcal{H}_{\Lambda}$, the canonical anticommutation relations (21) are equivalent to the commutation relations $D_{\xi}D_{\eta} - D_{\eta}D_{\xi} = -2i\omega(\xi, \eta)$ with the antisymmetric form (15) of the Weyl relations. Extending the operators (24) to \mathbb{R} -linear operators on \mathcal{H}_{Λ} by the rule

$$R(U, V)(\mu \otimes f) = \mu \otimes Uf + \mu^{*} \otimes Vf^{*}, \quad \mu \in \Lambda_1, f \in \mathcal{H}, \quad (26)$$

we obtain the relation

$$\omega(R\xi, R\eta) = \omega(\xi, \eta), \quad \xi, \eta \in \mathcal{H}_{\Lambda}, \quad (27)$$

which is equivalent to (25). Hence a unitary operator $T(R)$ on $\mathcal{A}(\mathcal{H})$ generates the Bogoliubov transformation (22) if

$$\hat{T}(R)W(\xi) = W(R\xi)\hat{T}(R) \quad (28)$$

is true with $\hat{T}(R) = \kappa_0 \otimes T(R)$ for all $\xi \in \mathcal{H}_{\Lambda}$.

4 The orthogonal group

4.1 Definition

The following proposition is well known:

Proposition 1 *The transformation (24) $\mathcal{H}_{\mathbb{R}} \ni f \rightarrow R(U, V)f = Uf + Vf^{*} \in \mathcal{H}_{\mathbb{R}}$ is an invertible linear transformation that preserves the inner product $(f \parallel g)_{\mathbb{R}} = \text{Re}(f \mid g)$ of $\mathcal{H}_{\mathbb{R}}$ if and only if U and V satisfy the identities*

$$UU^{\dagger} + VV^{\dagger} = I = U^{\dagger}U + V^T\bar{V}, \quad (29)$$

$$UV^T + VU^T = 0 = U^{\dagger}V + V^T\bar{U}. \quad (30)$$

These transformations form the group of orthogonal transformations, which will be denoted by $\mathcal{O}(\mathcal{H}_{\mathbb{R}})$. The multiplication rule is

$$R(U_2, V_2)R(U_1, V_1) = R(U_2U_1 + V_2\bar{V}_1, U_2V_1 + V_2\bar{U}_1). \quad (31)$$

The identity of the group is $R(I, 0)$, and the inverse of $R(U, V)$ is

$$R^{-1}(U, V) = R(U^{\dagger}, V^T). \quad (32)$$

In order to derive a unitary representation of the orthogonal group on the Fock space $\mathcal{A}(\mathcal{H})$ an additional constraint is necessary if $\dim \mathcal{H}$ is infinite: The operator V has to be a

HS operator [12, 27, 4]. The subset of all transformations $R(U, V)$ with a HS operator V forms a subgroup, which is denoted as $\mathcal{O}_2(\mathcal{H}_{\mathbb{R}})$. In the sequel we only consider transformations with this restriction, and “orthogonal transformation” always means a transformation in $\mathcal{O}_2(\mathcal{H}_{\mathbb{R}})$. The relations (29) and the HS condition for V imply

1. U and U^\dagger are Fredholm operators, i.e. the image spaces $\text{ran } U$ and $\text{ran } U^\dagger$ closed, and the spaces $\ker U$ and $\ker U^\dagger$ have finite dimension [13].
2. The spaces $\ker U$ and $\ker U^\dagger$ have the same dimension

$$\dim(\ker U) = \dim(\ker U^\dagger) < \infty. \quad (33)$$

The second statement follows from the identities (29), which imply $\ker U = \{f \mid V^T \bar{V} f = f\}$ and $\ker U^\dagger = \{f \mid V V^\dagger f = f\}$. The positive trace class operators $V V^\dagger$ and $V^T \bar{V}$ have coinciding eigenvalues within the interval $0 < \lambda \leq 1$, and the corresponding eigenspaces have the same finite dimension.

A suitable topology for $\mathcal{O}_2(\mathcal{H}_{\mathbb{R}})$ is given by the norm

$$\|R(U, V)\| := \|U\| + \|V\|_2, \quad (34)$$

where $\|U\|$ is the operator norm of $U \in \mathcal{L}(\mathcal{H})$ and $\|V\|_2$ is the HS norm of $V \in \mathcal{L}_2(\mathcal{H})$. With this topology $\mathcal{O}_2(\mathcal{H}_{\mathbb{R}})$ is a topological group, which has two connected components, cf. Refs. [3] § 6(1), [21] section 2.4. Thereby the identity component of $\mathcal{O}_2(\mathcal{H}_{\mathbb{R}})$ is given by all transformations for which the space $\ker U$ has an even dimension.

The subset of all operators $R(S, 0)$ with a unitary operator $S \in \mathcal{U}(\mathcal{H})$ forms the subgroup of all \mathbb{C} -linear orthogonal transformations, which is called \mathcal{Q} in the sequel.

4.2 Orthogonal transformations $R(U, V)$ with $\ker U \neq \{0\}$

If the operator U is not invertible, the Hilbert space can be decomposed into the orthogonal sums $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 = \mathcal{F}_0 \oplus \mathcal{F}_1$ of the following closed subspaces $\mathcal{H}_0 = \ker U^\dagger$, $\mathcal{H}_1 = \text{ran } U = (\ker U^\dagger)^\perp$ and $\mathcal{F}_0 = \ker U$, $\mathcal{F}_1 = \text{ran } U^\dagger = (\ker U)^\perp$. The projection operators onto \mathcal{H}_k are called P_k , and the projection operators onto \mathcal{F}_k are called Q_k , $k = 0, 1$. The identities (29) imply the relations $P_0 V V^\dagger = V V^\dagger P_0 = P_0$ and $\bar{Q}_0 V^\dagger V = V^\dagger V \bar{Q}_0 = \bar{Q}_0$ and the operator norm inequalities $\|V \bar{Q}_1\| < 1$ and $\|V^\dagger P_1\| < 1$. From the identities (30) we obtain the inclusions

$$V \mathcal{F}_1^* \subset \mathcal{H}_1, \quad V^T \mathcal{H}_1^* \subset \mathcal{F}_1. \quad (35)$$

Therefore $P_0 V = V \bar{Q}_0$ is an isometric mapping from \mathcal{F}_0^* onto \mathcal{H}_0 , and $\bar{Q}_0 V^\dagger = V^\dagger P_0$ is the inverse isometry from \mathcal{H}_0 onto \mathcal{F}_0^* .

The orthogonal transformation (24) $R(U, V)$ is the sum of two \mathbb{R} -linear mappings

$$R(U, V) = R(0, P_0 V) + R(U, P_1 V). \quad (36)$$

The transformation $R(0, P_0 V) = R(0, V \bar{Q}_0)$ is an isometric mapping from $\mathcal{F}_{0\mathbb{R}}$ onto $\mathcal{H}_{0\mathbb{R}}$ with inverse $R(0, Q_0 V^T)$, and $R(U, P_1 V) = R(P_1 U, V \bar{Q}_1)$ is an isometric mapping from $\mathcal{F}_{1\mathbb{R}}$ onto $\mathcal{H}_{1\mathbb{R}}$ with inverse $R(U^\dagger, Q_1 V^T)$, cf. the identity (32).

4.3 The left coset space $\mathcal{O}_2/\mathcal{Q}$

The left coset space $\mathcal{O}_2/\mathcal{Q}$ is the space of all orbits

$\{R(U, V)R(S, 0) = R(US, V\bar{S}) \mid S \in \mathcal{U}(\mathcal{H})\}$ under right action of the subgroup \mathcal{Q} . This space can serve to identify states in the Fock space representation of the orthogonal group. If U is invertible the operator $U^{\dagger-1}V^T = U^{\dagger-1}SS^{\dagger}V^T = (US)^{\dagger-1}(V\bar{S})^T$ is invariant against the right action of this subgroup. From the identity (30) we obtain $U^{\dagger-1}V^T + V\bar{U}^{-1} = U^{\dagger-1}(V^T\bar{U} + U^{\dagger}V)\bar{U}^{-1} = 0$. Hence

$$X := V\bar{U}^{-1} = -U^{\dagger-1}V^T = -X^T \quad (37)$$

is a skew symmetric HS operator, which characterizes the orbit through $R(U, V)$.

Remark 3 *Knowing the operator (37) one can easily obtain a transformation on the orbit. The operators $L = (I + XX^{\dagger})^{-\frac{1}{2}} \geq 0$ and $W = X(I + X^{\dagger}X)^{-\frac{1}{2}} = -W^T$ satisfy the conditions (29) and (30) so that $R(L, W)$ is an orthogonal transformation. The formula (37) $W\bar{L}^{-1} = X$ reproduces the input X . Hence $R(L, W)$ is a point on the orbit specified by X .*

If $\ker U \neq 0$ the space $\mathcal{H}_0 = \ker U^{\dagger} = \ker (US)^{\dagger}$, S unitary, is invariant against the right action of the group \mathcal{Q} . This space can therefore be used as a coordinate for an orbit. But the operator (37) is no longer defined. It is convenient to introduce the notion of a *generalized inverse operator*. Let $A \in \mathcal{L}(\mathcal{H})$ be an operator with closed range. Then the generalized inverse $A^{(-1)} \in \mathcal{L}(\mathcal{H})$ of A is defined as, cf. e. g. [14, 6],

$$A^{(-1)}f := A^{-1}Pf = \begin{cases} A^{-1}f & \text{if } f \in \text{ran } A, \\ 0 & \text{if } f \in \ker A^{\dagger} = (\text{ran } A)^{\perp}, \end{cases} \quad (38)$$

where P is the projector onto $\text{ran } A$. The operator $A^{(-1)}$ satisfies the identities $AA^{(-1)} = P$ and $A^{(-1)}A = Q$, thereby Q is the projector onto $\text{ran } A^{\dagger}$. The relations $(A^{\dagger})^{(-1)} = (A^{(-1)})^{\dagger}$, $(\bar{A})^{(-1)} = \overline{(A^{(-1)})}$ and $(A^T)^{(-1)} = (A^{(-1)})^T$ are valid. In the sequel we write $A^{\dagger(-1)}$ instead of $(A^{\dagger})^{(-1)}$. If S is a unitary operator, then the operators SA and AS have also a closed range, and the following identities are true

$$(SA)^{(-1)} = A^{(-1)}S^{\dagger}, \quad (AS)^{(-1)} = S^{\dagger}A^{(-1)}. \quad (39)$$

Using the notation of the definition (38) the generalization of (37) is the HS operator

$$X := V\bar{U}^{(-1)} = V\bar{U}^{-1}\bar{P}_1 \stackrel{(35)}{=} P_1V\bar{U}^{-1}\bar{P}_1 = -V^TU^{\dagger(-1)}. \quad (40)$$

The skew symmetry is again a consequence of (30): $X + X^T = P_1(V\bar{U}^{-1} + U^{\dagger-1}V^T)\bar{P}_1 = U^{\dagger(-1)}(U^{\dagger}V + V^T\bar{U})\bar{U}^{(-1)} \stackrel{(30)}{=} 0$. The operator (40) is invariant against right action of the subgroup \mathcal{Q} : $V\bar{U}^{(-1)} = V\bar{S}S^T\bar{U}^{(-1)} \stackrel{(39)}{=} (V\bar{S})\overline{(US)}^{(-1)}$, and it can serve as second coordinate for an orbit.

5 Representations

This section presents the central result of this paper: the construction of a representation of $\mathcal{O}_2(\mathcal{H}_{\mathbb{R}})$. We are looking for operators $T(R)$, which have the properties

$$R \in \mathcal{O}_2(\mathcal{H}_{\mathbb{R}}) \rightarrow T(R) \text{ unitary operator on } \mathcal{A}(\mathcal{H}), \quad T(id) = I_{\mathcal{A}(\mathcal{H})}, \quad (41)$$

and which implement canonical transformations, cf. (23),

$$T(R)\Delta(f)T^\dagger(R) = \Delta(Rf), \quad f \in \mathcal{H}_{\mathbb{R}}. \quad (42)$$

Since the CAR algebra generated by $\{\Delta(f) \mid f \in \mathcal{H}_{\mathbb{R}}\}$ is irreducible within $\mathcal{A}(\mathcal{H})$, the unitary operators $T(R)$ are determined by (41) up to a phase factor. Moreover, the group laws of $\mathcal{O}_2(\mathcal{H}_{\mathbb{R}})$ and the relation (41) imply $T(R_2)T(R_1)\Delta(f)T^\dagger(R_1)T^\dagger(R_2) = T(R_2)\Delta(R_1f)T^\dagger(R_2) = \Delta(R_2R_1f)$, and the product rule

$$T(R_2)T(R_1) = \chi(R_2, R_1)T(R_2R_1) \text{ with } \chi(R_2, R_1) \in \mathbb{C}, \quad |\chi(R_2, R_1)| = 1 \quad (43)$$

follows. Hence $R \in \mathcal{O}_2(\mathcal{H}_{\mathbb{R}}) \rightarrow T(R)$ is a unitary ray representation. The multiplier χ satisfies the cocycle identity $\chi(R_3, R_2)\chi(R_3R_2, R_1) = \chi(R_3, R_2R_1)\chi(R_2, R_1)$ as consequence the associativity of the operator product.

A simple class of canonical transformations arise if $R(U, V)$ is \mathbb{C} -linear, i.e. for $U = S \in \mathcal{U}(\mathcal{H})$ and $V = 0$. Then the condition (25) is fulfilled and $T(S, 0) = \Gamma(S)$ is a unitary operator on $\mathcal{A}(\mathcal{H})$ that leads to the transformation $Ta^\pm(f)T^+ = a^\pm(Sf)$ of the creation/annihilation operators in agreement with the rule (22). The group laws and (43) imply the relations

$$\Gamma(S)T(U, V) = \chi T(SU, SV), \quad T(U, V)\Gamma(S) = \chi T(US, V\bar{S}) \quad (44)$$

with phase factors χ .

In Sec. 5.1 the case of orthogonal transformations $R(U, V)$ with an invertible operator U is investigated. For these transformations superanalytic methods are adequate. The construction is first given for operators $\hat{T}(R) = \hat{T}[U, V]$ on the module space $\mathcal{A}^\Lambda(\mathcal{H})$. These operators have the structure $\hat{T}(R) = \kappa_0 \otimes T(R)$ with $T(R) \in \mathcal{L}(\mathcal{A}(\mathcal{H}))$, and the pull-back to operators $T(R)$ on the Fock space $\mathcal{A}(\mathcal{H})$ can be easily performed. The case of orthogonal transformations with operators U , which are not invertible, is treated in Sec. 5.2. Then the construction given in Sec. 5.1 is only possible on the subspace $\mathcal{A}^\Lambda(\mathcal{F}_1)$, $\mathcal{F}_1 = \text{ran } U^\dagger$, and a separate investigation concerning the finite dimensional space $\mathcal{A}(\mathcal{F}_0)$, $\mathcal{F}_0 = \ker U$, is necessary.

5.1 Transformations $R(U, V)$ with invertible U

5.1.1 The ansatz

Let $R(U, V)$ be an orthogonal transformation of the group $\mathcal{O}_2(\mathcal{H}_{\mathbb{R}})$ with an invertible operator U . In correspondence to the case of bosons – cf. Sec. 5.1. in [18] – the representation $\hat{T}(R) = \hat{T}[U, V]$ of this transformation is now defined on the set of exponential vectors $\exp \xi$, $\xi \in \mathcal{H}_\Lambda$, by the ansatz

$$\hat{T}[U, V] \exp \xi := c_X \exp \left(-\frac{1}{2} \left\langle \xi \parallel V^\dagger U^{\dagger-1} \xi \right\rangle \right) \Psi(X, U^{\dagger-1} \xi) \in \mathcal{C}^\Lambda(\mathcal{H}_\Lambda) \quad (45)$$

with the skew symmetric HS operator (37) $X = V\bar{U}^{-1} = -U^{\dagger-1}V^T$ and the normalization constant

$$c_X = \left(\det \left(I + X^\dagger X \right) \right)^{-1/4} = c_{X^\dagger}. \quad (46)$$

The tensor Ψ is an ultracoherent vector (10). The operator $Y := -V^\dagger U^{\dagger-1} = \bar{U}^{-1}\bar{V}$ is a skew symmetric HS operator, and the quadratic form $\langle \xi \parallel Y \xi \rangle$ is an element of Λ_2 . The exponential series $\exp \left(\frac{1}{2} \langle \xi \parallel Y \xi \rangle \right)$ converges within $\Lambda_{\bar{0}}$, cf. Appendix A.2. The right side of (45) is therefore a well defined element of $\mathcal{C}^\Lambda(\mathcal{H}_\Lambda)$. The exponential vector $\exp \xi$ and (45) are entire analytic functions in the variable $\xi \in \mathcal{H}_\Lambda$. Using (45) and (11) we derive the identity

$$\begin{aligned} \left(\exp \xi \mid \hat{T}(R) \exp \eta \right) &= c_X \left(\exp \xi \mid \Psi(V\bar{U}^{-1}, U^{\dagger-1}\eta) \exp \left(-\frac{1}{2} \langle \eta \parallel V^\dagger U^{\dagger-1} \eta \rangle \right) \right) = \\ c_X \exp \left(-\frac{1}{2} \langle \eta \parallel V^\dagger U^{\dagger-1} \eta \rangle \right) \exp \left(U^{-1} \xi \mid \eta - \frac{1}{2} V^T \xi^* \right) &= \left(\hat{T}(R^{-1}) \exp \xi \mid \exp \eta \right) \end{aligned} \quad (47)$$

with $R^{-1}(U, V) = R(U^\dagger, V^T)$, cf. (32). Since ξ and η are arbitrary vectors in \mathcal{H}_Λ , the identity (47) implies that (45) has a Λ -linear extension onto $\mathcal{C}^\Lambda(\mathcal{H}_\Lambda)$ that satisfies $\hat{T}(R)(\lambda \exp \xi) = \lambda \left(\hat{T}(R) \exp \xi \right)$, $\lambda \in \Lambda$. Moreover, we obtain the operator identity

$$\hat{T}^+(R) \equiv \left(\hat{T}(R) \right)^+ = \hat{T}(R^{-1}) \quad (48)$$

on the space $\mathcal{C}^\Lambda(\mathcal{H}_\Lambda)$. Using the formula (12) the Λ -inner product $\left(\hat{T}(R) \exp \xi \mid \hat{T}(R) \exp \xi \right)$ is calculated with the result $\left(\hat{T}(R) \exp \xi \mid \hat{T}(R) \exp \xi \right) = \exp(\xi \mid \xi)$ for all $R \in \mathcal{O}_2(\mathcal{H}_\mathbb{R})$ and $\xi \in \mathcal{H}_\Lambda$. By analyticity arguments this identity implies $\left(\hat{T}(R) \exp \xi \mid \hat{T}(R) \exp \eta \right) = \left(\exp \xi \mid \hat{T}^+(R) \hat{T}(R) \exp \eta \right) = (\exp \xi \mid \exp \eta)$ for all $\xi, \eta \in \mathcal{H}_\Lambda$, and we obtain the operator identity

$$\hat{T}^+(R) \hat{T}(R) = id \quad (49)$$

on $\mathcal{C}^\Lambda(\mathcal{H}_\Lambda)$. The relations (48) and (49), which are valid for all $R \in \mathcal{O}_2(\mathcal{H}_\mathbb{R})$, imply that $\hat{T}(R)$ is an invertible operator on $\mathcal{C}^\Lambda(\mathcal{H})$ with inverse $\left(\hat{T}(R) \right)^{-1} = \hat{T}^+(R)$.

To obtain the pull-back of the operators $\hat{T}(R)$ to the Fock space $\mathcal{A}(\mathcal{H})$ we observe that the ansatz (45) has the following structure: $\hat{T}(R)$ does not operate on the Λ factors of $\exp \xi$, hence $\hat{T}(R)$ is the product

$$\hat{T}(R) = \kappa_0 \otimes T(R), \quad (50)$$

where $T(R)$ is an operator on $\mathcal{A}(\mathcal{H})$. The explicit form of $T(R)$ can be recovered from (45) by contraction of the Λ -factors. For any $\lambda \in \Lambda$ we have $T(R)(\lambda \mid \exp \xi)_\Lambda = \left(\lambda \mid \hat{T}(R) \exp \xi \right)_\Lambda$. Since the \mathbb{C} -linear span of the set $\{(\lambda \mid \exp \xi)_\Lambda\}$ with $\lambda \in \Lambda$ and $\xi \in \mathcal{H}_\Lambda$ is dense in $\mathcal{A}(\mathcal{H})$, the operator $T(R)$ is determined by (45). The factorization (50) and the identities (48) and (49) imply that the operator $T(R)$ has a unique extension to a unitary operator on $\mathcal{A}(\mathcal{H})$.

Remark 4 Starting from the expansion $\xi = \sum_{m \in \mathbb{N}} \kappa_m \otimes f_m \in \mathcal{H}_\Lambda$ with an ON basis $\{\kappa_m\}$ of Λ_1 and vectors $f_m \in \mathcal{H}$, one obtains a more explicit version of the ansatz (45), which allows to derive $T(R)(f_1 \wedge \dots \wedge f_n)$ for arbitrary vectors $f_j \in \mathcal{H}$, $j = 1, \dots, n$, $n \in \mathbb{N}$. The calculation is given in the Appendix B.1.

The formula (45) applies to transformations $R(U, 0)$ with a unitary operator $U = S \in \mathcal{U}(\mathcal{H})$. For these transformations it has the simple form $\hat{T}[S, 0] \exp \xi = \exp(S\xi)$, and $\hat{T}[S, 0]$ is identified with $\hat{T}[S, 0] = \hat{\Gamma}(S) := \kappa_0 \otimes \Gamma(S)$. The unitary transformation $\Gamma(S)$ is a canonical transformation on $\mathcal{A}(\mathcal{H})$ as already stated in the introductory part of Sec. 5. The operators (45) satisfy the relations

$$\hat{\Gamma}(S)\hat{T}[U, V] = \hat{T}[SU, SV] \text{ and } \hat{T}[U, V]\hat{\Gamma}(S) = \hat{T}[US, V\bar{S}], \quad (51)$$

which imply the corresponding identities for $T[U, V]$.

If U is not invertible, the ansatz (45) is not defined for vectors $\xi \in \mathcal{H}_\Lambda = \Lambda_1 \hat{\otimes} \mathcal{H}$ with a component in $\Lambda_1 \hat{\otimes} \mathcal{F}_0$, $\mathcal{F}_0 = \ker U$. But we can obtain a mapping $\hat{T}_1 = \hat{T}_1[U, P_1 V] = \hat{T}_1(R_1)$ from $\mathcal{C}^\Lambda(\Lambda_1 \hat{\otimes} \mathcal{F}_1) = \hat{\Gamma}(Q_1)\mathcal{C}^\Lambda(\mathcal{H}_\Lambda)$ onto $\mathcal{C}^\Lambda(\Lambda_1 \hat{\otimes} \mathcal{H}_1) = \hat{\Gamma}(P_1)\mathcal{C}^\Lambda(\mathcal{H}_\Lambda)$ that represents the transformation $R_1 = R(U, P_1 V)$ of Eq. (36).² On the restricted set of coherent vectors $\exp \xi$ with $\xi \in \Lambda_1 \hat{\otimes} \mathcal{F}_1$, $\mathcal{F}_1 = \text{ran } U^\dagger$, we define the operator

$$\hat{T}_1[U, P_1 V] \exp \xi := c_X \Psi(X, U^{\dagger(-1)}\xi) \exp \left(-\frac{1}{2} \langle \xi \parallel V^\dagger U^{\dagger(-1)}\xi \rangle \right) \in \mathcal{C}^\Lambda(\Lambda_1 \hat{\otimes} \mathcal{H}_1) \quad (52)$$

with the skew symmetric operator (40) X and the normalization constant (46). The right side of (52) depends only on the operators $U = P_1 U$ and $P_1 V$. Arguments as used for (45) yield that the ansatz (52) can be extended to an invertible mapping from $\mathcal{A}^\Lambda(\mathcal{F}_1)$ onto $\mathcal{A}^\Lambda(\mathcal{H}_1)$.

There is another approach to the operator \hat{T}_1 . Let U_0 be a partial isometry from \mathcal{F}_0 onto \mathcal{H}_0 , i.e. the operator U_0 satisfies $U_0 U_0^\dagger = P_0$ and $U_0^\dagger U_0 = Q_0$. Then the transformation $R(U + U_0, P_1 V)$ is an orthogonal transformation with invertible first argument $(U + U_0)^{-1} = U^{(-1)} + U_0^\dagger$, and the operator $\hat{T}[U + U_0, P_1 V]$ can be defined by the ansatz (45). The mapping (52) is the restriction of the operator $\hat{T}[U + U_0, P_1 V]$ to the space $\mathcal{A}^\Lambda(\mathcal{F}_1) = \hat{\Gamma}(Q_1)\mathcal{A}^\Lambda(\mathcal{H})$

$$\hat{T}_1[U, P_1 V] \hat{\Gamma}(Q_1) = \hat{T}[U + U_0, P_1 V] \hat{\Gamma}(Q_1). \quad (53)$$

If S is a unitary operator on \mathcal{H} , we infer from (52), or from (51) and (53), the following identity, which is needed in Sec. 5.2.2,

$$\hat{T}_1[U, P_1 V] \hat{\Gamma}(Q_1) = \hat{T}_1[US, P_1 V\bar{S}] \hat{\Gamma}(S^\dagger Q_1). \quad (54)$$

As a consequence of (50) and (53) the operator \hat{T}_1 has the structure $\hat{T}_1(R_1) = \kappa_0 \otimes T_1(R_1)$, where T_1 is an isometric surjective mapping from $\mathcal{A}(\mathcal{F}_1)$ onto $\mathcal{A}(\mathcal{H}_1)$.

5.1.2 The intertwining relation with Weyl operators

The ansatz (45) and the relation (13) for the Weyl operators yield the identity

$$\begin{aligned} \hat{T}(R)W(\xi) \exp \eta &= c_X \Psi \left(X, U^{\dagger-1}(\xi + \eta) \right) \times \\ &\exp \left(-(\xi \mid \eta) - \frac{1}{2} (\xi \mid \xi) - \frac{1}{2} \langle \xi + \eta \parallel V^\dagger U^{\dagger-1}(\xi + \eta) \rangle \right) \end{aligned} \quad (55)$$

with the operator (37) X , the normalization constant (46), and supervectors $\xi, \eta \in \mathcal{H}_\Lambda$. On the other hand $W(R\xi)\hat{T}(R) \exp \eta$ is calculated using (18), (29), (30) and (45) with the same result. The intertwining relation

$$\hat{T}(R)W(\xi) = W(R\xi)\hat{T}(R) \quad (56)$$

²For the definition of the spaces and the operators see Sec. 4.2.

is therefore true on the space $\mathcal{C}(\mathcal{H}_\Lambda)$ for all $R(U, V) \in \mathcal{O}_2(\mathcal{H}_\mathbb{R})$ with invertible U and for all supervectors $\xi \in \mathcal{H}_\Lambda$. Then Lemma 4 implies that this identity is true on the module Fock space $\mathcal{A}^\Lambda(\mathcal{H})$. Hence the operators (45) implement Bogoliubov transformations as defined in Sec. 3.2.

If U is not invertible, we can derive a partial intertwining relation for the mapping (52). The orthogonal transformation R is restricted to the isometric mapping $R_1 := R(U, P_1 V)$ from $\mathcal{F}_{1\mathbb{R}}$ onto $\mathcal{H}_{1\mathbb{R}}$. If η is a supervector in $\Lambda_1 \hat{\otimes} \mathcal{F}_1$, the tensor $R_1 \eta = U\eta + P_1 V \eta^*$ is a supervector in $\Lambda_1 \hat{\otimes} \mathcal{H}_1$. Then the calculations given above go through with supervectors $\xi, \eta \in \Lambda_1 \hat{\otimes} \mathcal{F}_1$ and the skew symmetric operator (40) X . The resulting identity $\hat{T}(R_1)W(\xi) \exp \eta = W(R_1 \xi) \hat{T}(R_1) \exp \eta$ implies the relation

$$\hat{T}_1(R_1)W(\xi)\hat{\Gamma}(Q_1) = W(R_1\xi)\hat{T}_1(R_1)\hat{\Gamma}(Q_1). \quad (57)$$

5.2 Transformations $R(U, V)$ with $\ker U \neq \{0\}$

If $\ker U = \mathcal{F}_0 \neq \{0\}$ it is convenient to split the orthogonal transformation $R(U, V)$ into the sum (36). At the end of Sec. 5.1.1 we have introduced the isometric mapping T_1 from $\mathcal{A}(\mathcal{F}_1)$ onto $\mathcal{A}(\mathcal{H}_1)$ as representation of the partial isometry $R_1 = R(U, P_1 V)$, the second term in the sum (36). In Sec. 5.2.1 we construct an isometric mapping T_0 from $\mathcal{A}(\mathcal{F}_0)$ onto $\mathcal{A}(\mathcal{H}_0)$ that is a representation of the partial isometry $R_0 = R(0, P_0 V)$, the first term in the sum (36). In Sec. 5.2.2 we derive that the combined action of these operators is a representation of the orthogonal transformation (36) on the Fock space $\mathcal{A}(\mathcal{H})$.

5.2.1 A duality mapping between finite dimensional spaces

We start with a linear partial isometry J from \mathcal{F}_0^* onto \mathcal{H}_0 , i.e. J satisfies the identities $J^\dagger J = \bar{Q}_0$ and $JJ^\dagger = P_0$. We choose an ON basis $\{e_m, m \in \mathbf{M}\}$, $\mathbf{M} = \{1, \dots, n\}$ of \mathcal{H}_0 . Then the vectors $\{f_m = -J^T e_m^* \mid m \in \mathbf{M}\}$ form an ON basis of \mathcal{F}_0 . We define the linear mapping $T_0[J]$ by its action on the basis $\{f_{\mathbf{K}}, \mathbf{K} \subset \mathbf{M}\}$ of $\mathcal{A}(\mathcal{F}_0)$

$$T_0[J] f_{\mathbf{K}} := (-1)^{\tau(\mathbf{K}, \mathbf{N})} e_{\mathbf{N} \setminus \mathbf{K}}, \quad \mathbf{K} \subset \mathbf{N}. \quad (58)$$

The number $\tau(\mathbf{K}, \mathbf{N})$ has been defined at the end of Sec. 2.1. The linear extension of (58) is obviously an isometric linear isomorphism $T_0[J]$ between $\mathcal{A}(\mathcal{F}_0)$ and $\mathcal{A}(\mathcal{H}_0)$. The operator T_0 has the following intertwining relations with the operator $\Delta(h) = a^+(h) - a^-(h)$

$$T_0[J] \Delta(h) = \Delta(R_0 h) T_0[J] \quad \text{if } h \in \mathcal{F}_0. \quad (59)$$

Thereby $R_0 := R(0, J)$ is the \mathbb{R} -linear operator (24) $R(0, J)h = Jh^*$. The proof of these relations is given in Appendix B.2. The irreducibility of the CAR algebras of the spaces $\mathcal{A}(\mathcal{F}_0)$ and $\mathcal{A}(\mathcal{H}_0)$ implies that the operator T_0 is unique except for a phase factor. Hence the constructions of the operator (58) based on different choices of the basis $\{e_j\}$ can disagree only in a multiplicative phase factor.

Remark 5 In Ref. [26] the mapping (58) has been constructed using creation and annihilation operators. If $\mathcal{F}_0 = \mathcal{H}_0 = \mathcal{H}_0^* - e.$ g. in the case of a real self-adjoint operator U – the mapping (58) is a modified Hodge star operator within $\mathcal{A}(\mathcal{H}_0)$.

The ambiguity of $T_0[J]$ is an overall phase factor, which can be fixed at the vacuum f_\emptyset of the space \mathcal{F}_0 . With the labels $T_0[J, \hat{e}]$, where $\hat{e} = T_0[J, \hat{e}] f_\emptyset \in \mathcal{H}_0^{\wedge n}$ is the image of the vacuum vector, the mapping $T_0[J, \hat{e}]$ is uniquely determined. With this phase convention the relation

$$T_0[J, \hat{e}] \Gamma(S) = T_0[J\bar{S}, \hat{e}] \quad (60)$$

is valid for all unitary operators $S \in \mathcal{U}(\mathcal{H})$.

The mapping $T_0[J]$ has the natural extension $\hat{T}_0[J] = \kappa_0 \otimes T_0[J]$ to an invertible mapping from $\Lambda \hat{\otimes} \mathcal{A}(\mathcal{F}_0)$ onto $\Lambda \hat{\otimes} \mathcal{A}(\mathcal{H}_0)$. The intertwining relation of this operator $\hat{T}_0(R_0) = \hat{T}_0[J]$ with the Weyl operator follows from (59) as

$$\hat{T}_0(R_0)W(\xi)\hat{\Gamma}(Q_0) = W(R_0\xi)\hat{T}_0(R_0)\hat{\Gamma}(Q_0). \quad (61)$$

Thereby $\hat{\Gamma}(Q_0)$ is the projection operator onto the space $\Lambda \hat{\otimes} \mathcal{A}(\mathcal{F}_0) \subset \mathcal{A}^\Lambda(\mathcal{H})$.

There is a structural difference between operators $\hat{T}_0[J]$ with an even dimension of $\mathcal{H}_0 = \text{ran } J (= \ker U^\dagger)$ and operators with an odd dimension of this space. If $\dim \mathcal{H}_0$ is even, the operator \hat{T}_0 maps tensors of parity π onto tensors with the same parity. If $\dim \mathcal{H}_0$ is odd, the operator \hat{T}_0 changes the parity π into $\pi + 1 \bmod 2$. This structural difference of $\hat{T}_0[J]$ reflects the group theoretical difference between the orthogonal transformations $R(U, V)$ with $\dim(\ker U)$ even or odd. If $\dim(\ker U)$ is even (odd), the orthogonal transformation $R(U, V)$ belongs (does not belong) to the connectivity component of the group identity, cf. § 6 of [2].

5.2.2 The general case

The canonical transformation $\hat{T}(R)$ is defined on $\mathcal{A}^\Lambda(\mathcal{H})$ combining the operators \hat{T}_0 and (52) \hat{T}_1 . Thereby the operator J is identified with $P_0V = V\bar{Q}_0$. We have to take into account that \hat{T}_0 interchanges the parity of states, if $n = \dim \mathcal{F}_0 = \dim \mathcal{H}_0$ is an odd number. The mapping $\hat{T}(R) = \hat{T}[U, V]$ is now defined on the product $\Xi = \Xi_0 \circ \Xi_1 \in \mathcal{A}^\Lambda(\mathcal{H})$ of tensors $\Xi_k \in \mathcal{A}^\Lambda(\mathcal{F}_k) = \hat{\Gamma}(Q_k)\mathcal{A}^\Lambda(\mathcal{H})$, $k = 0, 1$, as

$$\hat{T}(R)(\Xi_0 \circ \Xi_1) = \left(\hat{T}_0(R_0)\Xi_0 \right) \circ \left(\hat{T}_1(R_1)\hat{\Gamma}((-1)^n Q_1)\Xi_1 \right) \quad (62)$$

with $n = \dim \mathcal{F}_0 = \dim \mathcal{H}_0$. The spaces \mathcal{F}_k and \mathcal{H}_k , $k = 0, 1$, and their projection operators are defined in Sec. 4.2. The mapping (62) is the Λ -extension of the operator $T(R)$ on $\mathcal{A}(\mathcal{H})$ that is constructed from T_0 and T_1 by

$$T(R)(F_0 \wedge F_1) = (T_0(R_0)F_0) \wedge (T_1(R_1)\Gamma((-1)^n Q_1)F_1) \quad (63)$$

with tensors $F_k \in \mathcal{A}(\mathcal{F}_k)$, $k = 0, 1$. From the Sects. 5.1 and 5.2.1 we know that the operators T_k , $k = 0, 1$, are surjective isometric mappings from $\mathcal{A}(\mathcal{F}_k)$ onto $\mathcal{A}(\mathcal{H}_k)$. The operator $\Gamma((-1)^n Q_1)$ is an invertible isometry on $\mathcal{A}(\mathcal{F}_1)$. Hence the linear extension of (63) is a unitary operator on $\mathcal{A}(\mathcal{H})$. If we fix the phase of the operator $T_0(R_0) = T_0[P_0V, \hat{e}]$ as done in Sec. 5.2.1, the definition (62) and the relations (54) and (60) imply the identity

$$T[U, V]\Gamma(S) = T[US, V\bar{S}] \quad (64)$$

without additional phase factor for all $S \in \mathcal{U}(\mathcal{H})$.

In the last step we give a proof that the operators (63) satisfy the intertwining relation (28) with the Weyl operators. Let $\Xi = \Xi_0 \circ \Xi_1 \in \mathcal{A}^\Lambda(\mathcal{H})$ be the product of tensors

$\Xi_k \in \mathcal{A}^\Lambda(\mathcal{F}_k)$, $k = 0, 1$, where Ξ_0 has the parity $\pi(\Xi_0) = p$. Then the identities (20) and (62) imply $\hat{T}(R)W(\xi)(\Xi_0 \circ \Xi_1) = \left(\hat{T}_0(R_0)W(Q_0\xi)\Xi_0\right) \circ \left(\hat{T}_1(R_1)\hat{\Gamma}((-1)^n Q_1)W((-1)^p Q_1\xi)\Xi_1\right)$. From (19) we know $\hat{\Gamma}((-1)^n Q_1)W((-1)^p Q_1\xi)\Xi_1 = W((-1)^{p+n} Q_1\xi)\hat{\Gamma}((-1)^n Q_1)\Xi_1$. Using the relations (57) and (61) we obtain

$$\begin{aligned} \hat{T}(R)W(\xi)(\Xi_0 \circ \Xi_1) = \\ \left(W(R_0\xi)\hat{T}_0(R_0)\Xi_0\right) \circ \left(W((-1)^{p+n} R_1\xi)\hat{T}_1(R_1)\hat{\Gamma}((-1)^n Q_1)\Xi_1\right). \end{aligned} \quad (65)$$

On the other hand we have from (20) and (62)

$$\begin{aligned} W(R\xi)\hat{T}(R)(\Xi_0 \circ \Xi_1) = \\ \left(W(R_0\xi)\hat{T}_0(R_0)\Xi_0\right) \circ \left(W((-1)^q R_1\xi)\hat{T}_1(R_1)\hat{\Gamma}((-1)^n Q_1)\Xi_1\right). \end{aligned} \quad (66)$$

where q is the parity of $\hat{T}_0(R_0)\Xi_0$. If n is even, the operator \hat{T}_0 does not change the parity, and $q = p$ follows; if n is odd, the operator \hat{T}_0 changes the parity, and $q = p + 1 \bmod 2$ follows. In both cases we have $q = p + n \bmod 2$, and the tensors (65) and (66) agree. Hence the relation (28) is derived for all $R \in \mathcal{O}_2(\mathcal{H}_\mathbb{R})$ and all $\xi \in \mathcal{H}_\Lambda$.

5.3 The orbit of the vacuum

Starting from $\Xi = \kappa_0 \otimes 1_{vac}$ in formula (62) we obtain the orbit of the vacuum in the module space $\mathcal{A}^\Lambda(\mathcal{H})$. If U is invertible, we can take the definition (45) with $\xi = 0$. The pull-back of this formula to $\mathcal{A}(\mathcal{H})$ is easily seen as

$$\Phi[U, V] = \Theta(X) := \left(\det(I + X^\dagger X)\right)^{-1/4} \exp \Omega(X) \quad (67)$$

with $X = V\bar{U}^{-1}$. If $\ker U^\dagger = \mathcal{H}_0 \neq \{0\}$ the additional factor $T_0[P_0V]1_{vac} = \hat{e} = e_1 \wedge \dots \wedge e_n \in \mathcal{H}_0^{\wedge n}$ appears that is determined – except for a phase factor – by the subspace $\mathcal{H}_0 = P_0\mathcal{H}$

$$\Phi[U, V] = e_1 \wedge \dots \wedge e_n \wedge \Theta(X) \quad (68)$$

with $X = V\bar{U}^{(-1)}$. Since $\text{ran } X \subset \mathcal{H}_1 = \text{ran } U$ the tensor $\Theta(X)$ is an element of $\mathcal{A}(\mathcal{H}_1)$. The vectors (67) have a positive overlap with the vacuum $(1_{vac} | \Theta(X)) = (\det(I + XX^\dagger))^{-1/4} > 0$, whereas the vectors (68) are always orthogonal to the vacuum.

If U is invertible, the relation

$$\Phi[U, V] = \Phi[US, V\bar{S}], \quad S \in \mathcal{U}(\mathcal{H}), \quad (69)$$

follows for the vectors (67). If $\ker U^\dagger = \mathcal{H}_0 \neq \{0\}$, we can fix the phase of the operators $T_0(R_0) = T_0[P_0V, \hat{e}]$ as indicated in Sec. 5.2.1. Then the rule (64) implies the relation (69) also for the vectors (68). As a consequence of this phase convention the vectors Φ only depend on the variables of the coset space $\mathcal{O}_2/\mathcal{Q}$. Using the notations $T(R)$ and $\Phi(R)$ instead of $T[U, V]$ and $\Phi[U, V]$, we obtain from (43) the transformation rule for vectors on the orbit of the vacuum

$$T(R_2)\Phi(R_1) = \chi \Phi(R_2 R_1), \quad (70)$$

where $R_j \in \mathcal{O}_2$, $j = 1, 2$, are orthogonal transformations, and χ is a phase factor. If R_1 and $R_3 = R_2 R_1$ are orthogonal transformations with invertible U , the transformation rule (70)) gets the form

$$T[U_2, V_2] \Theta(X_1) = \chi \Theta(X_3) \quad (71)$$

with X_3 calculated from the product rule (31). We can choose the operators $U_1 = (I + X_1 X_1^+)^{-\frac{1}{2}}$ and $V_1 = X_1 (I + X_1^+ X_1)^{-\frac{1}{2}}$, cf. Remark 3 in Sec. 4.3. Then $U_3 = U_2 U_1 + V_2 \bar{V}_1$ and $V_3 = U_2 V_1 + V_2 \bar{U}_1$ yield the operator

$$X_3 = V_3 \bar{U}_3^{-1} = (U_2 X_1 + V_2) (\bar{U}_2 + \bar{V}_2 X_1)^{-1} \quad (72)$$

on the right side of (71). A formula of this type is given in Sec. 12.2 of [22] for the finite dimensional orthogonal group.

A Fock space calculations

A.1 Products

The following norm estimate for the exterior algebra is well known

$$\|F \wedge G\| \leq \sqrt{\frac{(p+q)!}{p!q!}} \|F\| \|G\| \text{ if } F \in \mathcal{A}_p(\mathcal{H}), G \in \mathcal{A}_q(\mathcal{H}), \quad (73)$$

where $\|\cdot\|$ is the standard Fock space norm (1) and $p, q \in \{0\} \cup \mathbb{N}$ are the degrees of the tensors. The modified norm (2) of the Grassmann algebra Λ leads to the estimates

$$\|\lambda \mu\|_\Lambda \leq \sqrt{\frac{p!q!}{(p+q)!}} \|\lambda\|_\Lambda \|\mu\|_\Lambda \leq \|\lambda\|_\Lambda \|\mu\|_\Lambda \text{ if } \lambda \in \Lambda_p, \mu \in \Lambda_q. \quad (74)$$

These stricter bounds imply that the Grassmann product is continuous with [19, 20]

$$\|\lambda_1 \lambda_2\|_\Lambda \leq \sqrt{3} \|\lambda_1\|_\Lambda \|\lambda_2\|_\Lambda \text{ for all } \lambda_{1,2} \in \Lambda. \quad (75)$$

If $\lambda \in \Lambda_1$ the estimate $\|\lambda \mu\|_\Lambda \leq \|\lambda\|_\Lambda \|\mu\|_\Lambda$ follows from (74) for all $\mu \in \Lambda$. The definition (2) of the norm implies that the product $\lambda_{\mathbf{K}}$ of vectors in the generating space $\lambda_k \in \Lambda_1$, $k \in \mathbf{K}$, $|\mathbf{K}| = p \in \mathbb{N}$, has a norm $\|\lambda_{\mathbf{K}}\|_\Lambda \leq (p!)^{-1} \prod_{k \in \mathbf{K}} \|\lambda_k\|_\Lambda$.

The product $\Theta \circ \Xi$ is not defined on the full module Fock space $\mathcal{A}^\Lambda(\mathcal{H})$, but it can be extended to a larger space than $\mathcal{A}_{fin}^\Lambda(\mathcal{H})$ by continuity arguments. Any tensor $\Theta \in \mathcal{A}_{fin}^\Lambda(\mathcal{H})$ can be decomposed into $\Theta = \sum_{p=0}^\infty \Theta_p$ with $\Theta \in \Lambda \hat{\otimes} \mathcal{A}_p(\mathcal{H})$. The product has the structure $\Theta_p \circ \Xi_q \in \Lambda \hat{\otimes} \mathcal{A}_{p+q}(\mathcal{H})$ if $\Theta_p \in \Lambda \hat{\otimes} \mathcal{A}_p(\mathcal{H})$ and $\Xi_q \in \Lambda \hat{\otimes} \mathcal{A}_q(\mathcal{H})$ with norm estimate

$$\|\Theta_p \circ \Xi_q\|_\otimes \leq \sqrt{3 \frac{(p+q)!}{p!q!}} \|\Theta_p\|_\otimes \|\Xi_q\|_\otimes. \quad (76)$$

We introduce the family of Hilbert norms $\|\Theta\|_{(\alpha)}^2 = \sum_{p=0}^\infty (p!)^\alpha \|\Theta_p\|_\otimes^2$ with a parameter $\alpha \geq 0$. The completion of $\mathcal{A}_{fin}^\Lambda(\mathcal{H})$ with respect to the norm $\|\cdot\|_{(\alpha)}$ is called $\mathcal{A}_{(\alpha)}^\Lambda(\mathcal{H})$. The inclusions $\mathcal{A}_{fin}^\Lambda(\mathcal{H}) \subset \mathcal{A}_{(\alpha)}^\Lambda(\mathcal{H}) \subset \mathcal{A}_{(0)}^\Lambda(\mathcal{H}) = \mathcal{A}^\Lambda(\mathcal{H})$, $\alpha \geq 0$, are obvious. Then Proposition 3 in Appendix A of [19] (or the results of [20]) imply the statement:

Lemma 5 *The product of $\mathcal{A}_{fin}^\Lambda(\mathcal{H})$ can be extended to a continuous mapping $\mathcal{A}_{(\alpha)}^\Lambda(\mathcal{H}) \times \mathcal{A}_{(\beta)}^\Lambda(\mathcal{H}) \longrightarrow \mathcal{A}^\Lambda(\mathcal{H})$ for spaces $\mathcal{A}_{(\alpha)}^\Lambda(\mathcal{H})$ and $\mathcal{A}_{(\beta)}^\Lambda(\mathcal{H})$ if $\alpha > 0$ and $\beta > 0$.*

Another extension of the product is formulated in **Lemma 2**, for which the proof is given here. We start with a tensor $\Theta \in \mathcal{A}_{fin}^\Lambda(\mathcal{H})$ and choose an ON basis $\{\kappa_m \mid m \in \mathbb{N}\}$ of Λ_1 . Then $\xi \in \mathcal{H}_\Lambda$ can be written as $\xi = \sum_{m \in \mathbf{M}} \kappa_m \otimes h_m$ with $h_m \in \mathcal{H}$, $\sum_{m \in \mathbf{M}} \|h_m\|^2 = \|\xi\|_\otimes^2$ and an index set $\mathbf{M} \subset \mathbb{N}$. The set $\{p! \kappa_{\mathbf{L}} \mid \mathbf{L} \in \mathcal{P}(\mathbb{N}), p = |\mathbf{L}|\}$ is an ON basis of Λ , and the tensor Θ can be expanded as $\Theta = \sum_{\mathbf{L} \in \mathcal{P}(\mathbb{N})} p! \kappa_{\mathbf{L}} \otimes F(\mathbf{L})$ with $F(\mathbf{L}) \in \mathcal{A}(\mathcal{H})$, $\sum_{\mathbf{L} \in \mathcal{P}(\mathbb{N})} \|F(\mathbf{L})\|^2 = \|\Theta\|_\otimes^2$. The tensor $\xi \circ \Theta = \sum_{m, \mathbf{L}} p! \kappa_m \kappa_{\mathbf{L}} \otimes (h_m \wedge F(\mathbf{L}))$ has the norm

$\|\xi \circ \Theta\|_\otimes^2 \leq \sum_{\mathbf{K} \in \mathcal{P}(\mathbb{N})} ((p+1)^{-1} \sum' \|h_m \wedge F(\mathbf{L})\|)^2$, where the sum \sum' extends over the $p+1$ pairs $\{(m, \mathbf{L}) \in \mathbb{N} \times \mathcal{P}(\mathbb{N}) \mid \{m\} \cup \mathbf{L} = \mathbf{K}, |\mathbf{K}| = p+1\}$. This inequality implies³

$\|\xi \circ \Theta\|_\otimes^2 \leq \sum_{m, \mathbf{N}} \|h_m \wedge F(\mathbf{N})\|^2 \leq \left(\sum_{n \in \mathbb{N}} \|h_n\|^2 \right) \cdot \sum_{\mathbf{L} \in \mathcal{P}(\mathbb{N})} \|F(\mathbf{L})\|^2 = \|\xi\|_\otimes^2 \|\Theta\|_\otimes^2$. Hence the operator norm estimate (3) is true for $\Theta \in \mathcal{A}_{fin}^\Lambda(\mathcal{H})$. The extension to tensors $\Theta \in \mathcal{A}^\Lambda(\mathcal{H})$ follows by continuity.

The inequality (3) implies that the creation operator $b^+(\xi)$ is a continuous operator on $\mathcal{A}^\Lambda(\mathcal{H})$ with operator norm $\|b^+(\xi)\| \leq \|\xi\|_\otimes$. Moreover, a slight modification of the proof yields that the annihilation operator $b^-(\xi) = \sum_{m \in \mathbf{M}} \kappa_m^* \otimes a^-(h_m)$ is also continuous with operator norm $\|b^-(\xi)\| \leq \|\xi^*\|_\otimes = \|\xi\|_\otimes$.

A.2 Exponentials of tensors of second degree

If X is an operator in $\mathcal{L}_2^-(\mathcal{H})$, the following three statements are valid:

1. There exists an ON system $\{e_m \in \mathcal{H} \mid m \in \mathbf{M} \cup (-\mathbf{M})\}$ with $\mathbf{M} = \{1, 2, \dots, M\}$ or $\mathbf{M} = \mathbb{N}$ and complex numbers $z_m \in \mathbb{C}$ with $|z_1| \geq |z_2| \geq \dots > 0$ so that the mapping $\mathcal{H} \ni f \rightarrow Xf \in \mathcal{H}$ has the representation

$$Xf = \sum_{m \in \mathbf{M}} z_m (e_m \langle e_{-m} \mid f \rangle - e_{-m} \langle e_m \mid f \rangle). \quad (77)$$

The numbers z_m are square-summable with $\sum_{m \in \mathbf{M}} |z_m|^2 = 2^{-1} \text{tr} X^\dagger X = 2^{-1} \|X\|_2^2$. Here $\langle f \mid g \rangle = (f^* \mid g)$ is the \mathbb{C} -bilinear symmetric form introduced in Sec. 2.1.

2. There exists exactly one tensor $\Omega(X) \in \mathcal{A}_2(\mathcal{H})$ such that the identities

$$\langle \Omega(X) \mid f \wedge g \rangle = \langle f \wedge g \mid \Omega(X) \rangle = \langle f \mid Xg \rangle = -\langle g \mid Xf \rangle \quad (78)$$

are true for all $f, g \in \mathcal{H}$. The mapping $\mathcal{L}_2^-(\mathcal{H}) \ni X \rightarrow \Omega(X) \in \mathcal{A}_2(\mathcal{H})$ is a linear and continuous isomorphism, the norms are related by $\|\Omega(X)\|_2^2 = 2^{-1} \|X\|_{HS}^2$.

3. The exponential series (8) converges absolutely, and the inner product of two of these exponentials is (9).

The first Statement is an immediate consequence of Lemma 4.1 in [26].

³Here we use the estimate $\left(\sum_{j=1}^n x_j \right)^2 \leq n \left(\sum_{j=1}^n x_j^2 \right)$ if $x_j \geq 0$, $j = 1, \dots, n$.

The identities (78) imply that $X \rightarrow \Omega(X)$ is a linear bijective mapping from $X \in \mathcal{F}^-(\mathcal{H})$ – the space of skew symmetric finite rank operators – onto $\Omega(X) \in \mathcal{H} \wedge \mathcal{H}$. Given $X \in \mathcal{F}^-(\mathcal{H})$ the operator X has the representation (77) with a finite index set \mathbf{M} . The tensor

$$\Omega(X) := \sum_{m \in \mathbf{M}} z_m e_{-m} \wedge e_m \quad (79)$$

is an element of $\mathcal{H} \wedge \mathcal{H}$ and it satisfies the identities (78). Then the norm identity $\|X\|_2^2 = 2 \sum_{a \in \mathbf{M}} |z_a|^2 = 2 \|\Omega(X)\|_2^2$ yields that the linear mapping $\mathcal{F}^-(\mathcal{H}) \ni X \rightarrow \Omega(X) \in \mathcal{A}_2(\mathcal{H})$ can be extended by continuity to $\mathcal{L}_2^-(\mathcal{H})$. The relation (78) is equivalent to $(f \wedge g | \Omega(X)) = (g | X f^*)$.

For the proof of Statement 3 we start from the representation (79) of the tensor $\Omega(X)$ with a finite or countable set $\mathbf{M} \subset \mathbb{N}$. Then the powers of Ω are calculated as $\Omega^{\wedge p}(X) = p! \sum_{\mathbf{A}, |\mathbf{A}|=p} z_{\mathbf{A}} e_{-\mathbf{A}} \wedge e_{\mathbf{A}}$. Thereby $\sum_{\mathbf{A}, |\mathbf{A}|=p}$ means summation over all subsets $\mathbf{A} \subset \mathbf{M}$ with $|\mathbf{A}| = p \geq 1$ elements. The exponential series is $\exp \Omega(X) = \sum_{\mathbf{A}} z_{\mathbf{A}} e_{-\mathbf{A}} \wedge e_{\mathbf{A}}$, where the sum extends over all finite subsets \mathbf{A} of \mathbf{M} , including $\mathbf{A} = \emptyset$. The inner product

$$(\Omega^{\wedge p}(X) | \Omega^{\wedge p}(X)) = (p!)^2 \sum_{\mathbf{A}, |\mathbf{A}|=p} |z_{a_1}|^2 \dots |z_{a_p}|^2 \leq p! \left(\sum_{n \in \mathbf{M}} |z_n|^2 \right)^p = p! \left(\frac{1}{2} \|X\|_2^2 \right)^p \quad (80)$$

yields the identities

$$\begin{aligned} \|\exp \Omega(X)\|^2 &= \sum_{p=0,1,2,\dots} \left(\sum_{n_1 < n_2 < \dots < n_p} |z_{n_1}|^2 \dots |z_{n_p}|^2 \right) = \prod_{n \in \mathbf{M}} (1 + |z_n|^2) = \\ &= \sqrt{\det(I + X^\dagger X)} = \sqrt{\det(I + X X^\dagger)} \leq \exp \left(\frac{1}{2} \|X\|_2^2 \right). \end{aligned} \quad (81)$$

Hence the exponential series (8) converges in norm within $\mathcal{A}(\mathcal{H})$ for all $X \in \mathcal{L}_2^-(\mathcal{H})$, and $\mathcal{L}_2^-(\mathcal{H}) \ni X \rightarrow \exp \Omega(X) \in \mathcal{A}(\mathcal{H})$ is an entire analytic function. The mapping $\mathcal{L}_2^-(\mathcal{H}) \times \mathcal{L}_2^-(\mathcal{H}) \ni (X, Y) \rightarrow \varphi(X, Y) := (\exp \Omega(X) | \exp \Omega(Y)) \in \mathbb{C}$ is antianalytic in X and analytic in Y . Hence the function $\varphi(X, Y)$ is uniquely determined by the values on the diagonal $X = Y$, and $\varphi(X, X) = \sqrt{\det(I + X^\dagger X)} = \sqrt{\det(I + X X^\dagger)}$ implies (9).

Since the left side of (9) is antianalytic in X and analytic in Y the function $\sqrt{\det(I + X^\dagger Y)}$ can be expanded in a power series of the variables X^\dagger and Y . An explicit form can be obtained from the left side of (9). This expansion is often formulated with pfaffians of the operators X^\dagger and Y restricted to final dimensional subspaces, cf. e.g. Sec. 12 of [22] or [16].

The exponential of a tensor $\lambda \in \Lambda_2$ converges within Λ with norm estimate

$$\|\exp \lambda\|_\Lambda^2 \leq 1 + \|\lambda\|_\Lambda^2 + (2!)^{-2} \|\lambda^2\|_\Lambda^2 + \dots \stackrel{(74)}{\leq} \sum_{p=0}^{\infty} (p!)^{-2} \|\lambda\|_\Lambda^{2p} \leq \exp \left(\|\lambda\|_\Lambda^2 \right).$$

A.3 Ultracoherent vectors

The mapping $\mathcal{A}(\mathcal{H}) \ni F \rightarrow \kappa_0 \otimes F \in \mathcal{A}^\Lambda(\mathcal{H})$ gives a natural embedding of the Fock space $\mathcal{A}(\mathcal{H})$ into the Λ -module $\mathcal{A}^\Lambda(\mathcal{H}) = \Lambda \hat{\otimes} \mathcal{A}(\mathcal{H})$. The tensor (8) can therefore be identified with the element $\Psi(X) := \kappa_0 \otimes \exp \Omega(X) = \exp(\kappa_0 \otimes \Omega(X))$ of $\mathcal{A}^\Lambda(\mathcal{H})$. The norm of $\Psi(X)$ is $\|\Psi(X)\|_\otimes = \|\exp \Omega(X)\| \stackrel{(81)}{=} (\det(I + X^\dagger X))^{1/4}$. If $\xi \in \mathcal{H}_\Lambda^{alg}$ is an element of the algebraic superspace and $X \in \mathcal{L}_2^-(\mathcal{H})$ is a finite rank operator, the exponential functions are finite sums, and the identities $(\exp \xi) \circ \Psi(X) = \Psi(X) \circ (\exp \xi) = \exp(\xi + \kappa_0 \otimes \Omega(X))$ follow by algebraic calculation. The norm estimate (3) implies that the products $(\exp \xi) \circ \Psi(X)$ and

$\Psi(X) \circ (\exp \xi)$ are defined with a norm $\|(\exp \xi) \circ \Psi(X)\|_{\otimes} \leq \sum_p (p!)^{-1} \|\xi^p \circ \Psi(X)\|_{\otimes} \leq \sum_p (p!)^{-1} \|\xi\|_{\otimes}^p \cdot \|\Psi(X)\|_{\otimes} = (\exp \|\xi\|_{\otimes}) \|\Psi(X)\|_{\otimes}$. Hence the ultracoherent vector

$$\Psi(X, \xi) := (\exp \xi) \circ \Psi(X) = \Psi(X) \circ (\exp \xi) \quad (82)$$

is a well defined element of $\mathcal{A}^\Lambda(\mathcal{H})$ for all $X \in \mathcal{L}_2^-(\mathcal{H})$ and $\xi \in \mathcal{H}_\Lambda$. Moreover, the inequalities (3) and (80) imply the estimate $\|\xi^p \circ (\kappa_0 \otimes \Omega^q)\|_{\otimes} \leq \sqrt{q!} \|\xi\|_{\otimes}^p \cdot \|\Omega\|^q$ for all integers $p, q \geq 0$. Hence the series $\exp(\xi + \kappa_0 \otimes \Omega(X))$ converges uniformly in ξ and X within the space $\mathcal{A}^\Lambda(\mathcal{H})$ with the norm estimate $\|\exp(\xi + \kappa_0 \otimes \Omega(X))\|_{\otimes} \leq \sum_{p,q \geq 0} (p!)^{-1} (q!)^{-\frac{1}{2}} \|\xi\|_{\otimes}^p \left(2^{-\frac{1}{2}} \|X\|_2\right)^q$. The tensor (82) therefore coincides with $\exp(\xi + \kappa_0 \otimes \Omega(X))$ for all $X \in \mathcal{L}_2^-(\mathcal{H})$ and $\xi \in \mathcal{H}_\Lambda$. The tensors $\exp \xi$, $\Psi(X)$ and $\Psi(X, \xi)$ have even parity.

Remark 6 *The existence of the products (82) can also be derived from Lemma 5. The power ξ^p is an element of $\Lambda \widehat{\otimes} \mathcal{A}_p(\mathcal{H})$ with norm $\|\xi^p\|_{\otimes}^2 \leq \|\xi\|_{\otimes}^{2p}$. Hence $\exp \xi$ is an element of the space $\mathcal{A}_{(\alpha)}^\Lambda(\mathcal{H})$ if $\alpha \in [0, 2)$. The tensor $\kappa_0 \otimes \Omega^q(X)$ is an element of $\Lambda \widehat{\otimes} \mathcal{A}_{2q}(\mathcal{H})$ with norm $\|\kappa_0 \otimes \Omega^q(X)\|_{\otimes}^2 \leq (q!) \|X\|_2^{2q}$, cf. (80). Hence $\Psi(X)$ is an element of the space $\mathcal{A}_{(\beta)}^\Lambda(\mathcal{H})$ if $\beta \in [0, 1/2)$. The conditions of Lemma 5 for the product (82) are therefore satisfied.*

The relation $(f \wedge g | \Omega(X)) = (g | X f^*)$, $f, g \in \mathcal{H}$, implies $(\xi \circ \xi | \kappa_0 \otimes \Omega(X)) = (\xi | X \xi^*)$ with $\xi \in \mathcal{H}_\Lambda$. Then the identity

$$(\exp \xi | \Psi(X)) = \exp \frac{1}{2} (\xi | X \xi^*) \quad (83)$$

follows by series expansion and repeated use of the identity (6). The relation (11) is a consequence of the identities (5), (6) and (83).

To derive the action of the Weyl operator on an ultracoherent vector $\Psi(X, \xi)$ we calculate with the variables $\xi, \eta \in \mathcal{H}_\Lambda$ and $X \in \mathcal{L}_2^-(\mathcal{H})$ using (13) and (11) $(\exp \xi | W(\eta) \Psi(X, \xi)) = (W^+(\eta) \exp \xi | \Psi(X, \xi)) = \exp(-\frac{1}{2} \langle \eta^* | \eta \rangle + \frac{1}{2} \langle \eta^* | X \eta^* - 2\xi \rangle) \langle \exp \xi^* | \exp(\xi + \eta - X \eta^* + \kappa_0 \otimes \Omega(X)) \rangle$. This identity implies the formula (18). The restriction of (18) to $\Psi(X)$ is

$$W(\eta) \Psi(X) = \exp\left(-\frac{1}{2} \langle \eta^* | \eta - X \eta \rangle\right) \Psi(X, \eta - X \eta^*). \quad (84)$$

Let ξ be a supervector in \mathcal{H}_Λ , then $\eta = (I + X X^\dagger)^{-1} \xi + X(I + X^\dagger X)^{-1} \xi^*$ is an element of \mathcal{H}_Λ , which satisfies $\eta - X \eta^* = \xi$. With this supervector ξ we obtain from (84)

$$W(\eta) \Psi(X) = \Psi(X, \xi) \exp\left(\frac{1}{2} \left\langle \xi | (I - X^\dagger X)^{-1} \xi^* + X^\dagger (I - X X^\dagger)^{-1} \xi \right\rangle\right). \quad (85)$$

The inner product $(W \Psi | W \Psi)$ is known as $(W(\eta) \Psi(X) | W(\eta) \Psi(X)) \stackrel{(17)}{=} (\Psi(X) | \Psi(X)) \stackrel{(81)}{=} \kappa_0 \otimes \sqrt{\det(I + X^\dagger X)}$. Substituting (85) into this identity we obtain formula (12).

B Calculations for Sec. 5

B.1 The operator $T(R)$ of Sec. 5.1

In this Appendix we calculate the image of the operator $T(R)$ on arbitrary factorizing tensors $f_{\mathbf{M}} \in \mathcal{A}_p(\mathcal{H})$ with $|\mathbf{M}| = M \in \mathbb{N}$. We start from the expansion $\xi = \sum_{m \in \mathbb{N}} \kappa_m \otimes f_m \in \mathcal{H}_\Lambda$ with

an ON basis $\{\kappa_m\}$ of Λ_1 and vectors $f_m \in \mathcal{H}$ with $\sum_m \|f_m\|^2 = \|\xi\|_\otimes^2$. The coherent vector is $\exp \xi = \sum_{\mathbf{M} \subset \mathbb{N}} \kappa_{\mathbf{M}} \otimes f_{\mathbf{M}}$. The quadratic form $\langle \xi | Y \xi \rangle$ agrees with $\langle \kappa_0 \otimes \Omega(Y) | \xi \circ \xi \rangle$, cf. Appendix A.3, and its exponential is, cf. (83),

$$\exp \left(\frac{1}{2} \langle \xi | Y \xi \rangle \right) = \langle \exp (\kappa_0 \otimes \Omega(Y)) | \exp \xi \rangle = \sum_{\mathbf{M} \subset \mathbb{N}} \varphi(\mathbf{M}) \kappa_{\mathbf{M}}$$

with $\varphi(\mathbf{M}) := \langle \exp \Omega(Y) | f_{\mathbf{M}} \rangle$. The numbers $\varphi(\mathbf{M})$ have the values $\varphi(\emptyset) = 1$, $\varphi(\mathbf{M}) = 0$ if $|\mathbf{M}|$ is odd, and

$$\varphi(\mathbf{M}) = (q!)^{-1} \langle \Omega^{\wedge q}(Y) | f_{\mathbf{M}} \rangle = \text{Pf} (\langle f_m | Y f_n \rangle_{\mathbf{M}})$$

if $|\mathbf{M}| = M = 2q$, $q \in \mathbb{N}$. Thereby $\langle f_m | Y f_n \rangle_{\mathbf{M}}$ is the skew symmetric $M \times M$ matrix $\{\langle f_m | Y f_n \rangle, m \in \mathbf{M}, n \in \mathbf{M}\}$, and Pf is the pfaffian of this matrix. The Λ -dependent factors of the right side of (45) are

$$\begin{aligned} & \exp \left(\frac{1}{2} \langle \xi | Y \xi \rangle \right) \exp (U^{\dagger-1} \xi) = \\ & \left(\sum_{\mathbf{K} \subset \mathbb{N}} \varphi(\mathbf{K}) \kappa_{\mathbf{K}} \right) \left(\sum_{\mathbf{L} \subset \mathbb{N}} \kappa_{\mathbf{L}} \otimes (U^{\dagger-1} f)_{\mathbf{L}} \right) = \\ & \sum_{\mathbf{M} \subset \mathbb{N}} \kappa_{\mathbf{M}} \otimes \left(\sum_{\mathbf{K} \cup \mathbf{L} = \mathbf{M}, \mathbf{K} \cap \mathbf{L} = \emptyset} (-1)^{\tau(\mathbf{K}, \mathbf{L})} \varphi(\mathbf{K}) (U^{\dagger-1} f)_{\mathbf{L}} \right). \end{aligned}$$

The sign factor comes from $\kappa_{\mathbf{K}} \kappa_{\mathbf{L}} = (-1)^{\tau(\mathbf{K}, \mathbf{L})} \kappa_{\mathbf{K} \cup \mathbf{L}}$. Including the factor $c_X \exp (\kappa_0 \otimes \Omega(X))$ the right side of the ansatz (45) gets the form $\sum_{\mathbf{M} \subset \mathbb{N}} \kappa_{\mathbf{M}} \otimes F(\mathbf{M})$, where the tensors $F(\mathbf{M}) \in \mathcal{A}(\mathcal{H})$ are given by

$$F(\mathbf{M}) = T(R) f_{\mathbf{M}} = c_X \left(\sum_{\mathbf{K} \cup \mathbf{L} = \mathbf{M}, \mathbf{K} \cap \mathbf{L} = \emptyset} (-1)^{\tau(\mathbf{K}, \mathbf{L})} \varphi(\mathbf{K}) (U^{\dagger-1} f)_{\mathbf{L}} \right) \wedge \exp \Omega(X).$$

For tensors of degree less than 3 we obtain $T(R) 1_{vac} = c_X \exp \Omega(X)$, $T(R) f = c_X (U^{\dagger-1} f) \wedge \exp \Omega(X)$, and $T(R) (f_1 \wedge f_2) = c_X ((U^{\dagger-1} f_1) \wedge (U^{\dagger-1} f_2)) + \langle f_1 | Y f_2 \rangle 1_{vac} \wedge \exp \Omega(X)$.

B.2 Calculations for the duality mapping

If \mathbf{A} and \mathbf{B} are finite subsets of \mathbb{N} and n is an element of \mathbb{N} , we define the following numbers: $\tau(n, \mathbf{B}) := \# \{b \in \mathbf{B} \text{ with } n > b\}$ and $\tau(\mathbf{A}, \mathbf{B}) := \# \{(a, b) \in \mathbf{A} \times \mathbf{B} \mid a > b\}$. Let \mathcal{H} be a Hilbert space with ON basis $\{e_n \mid n \in \mathbb{N}\}$. The creation/annihilation operators $a_n^\pm = a^\pm(e_n)$ are determined by their values on the ON basis vectors $\{e_{\mathbf{A}} \mid \mathbf{A} \subset \mathbb{N}\}$ of $\mathcal{A}(\mathcal{H})$:

$$\begin{aligned} a_n^+ e_{\mathbf{A}} &= 0 \text{ if } n \in \mathbf{A}, & a_n^+ e_{\mathbf{A}} &= (-1)^{\tau(n, \mathbf{A})} e_{\mathbf{A} \cup \{n\}} \text{ if } n \notin \mathbf{A}, \\ a_n^- e_{\mathbf{A}} &= (-1)^{\tau(n, \mathbf{A})} e_{\mathbf{A} \setminus \{n\}} \text{ if } n \in \mathbf{A}, & a_n^- e_{\mathbf{A}} &= 0 \text{ if } n \notin \mathbf{A}. \end{aligned} \quad (86)$$

Taking the definitions of spaces and operators from Sec. 5.2.1 the operator $\Theta \equiv T_0[J]$ is defined as

$$\Theta f_{\mathbf{K}} := (-1)^{\tau(\mathbf{K}, \mathbf{M})} e_{\bar{\mathbf{K}}}, \quad \mathbf{K} \subset \mathbf{M}. \quad (87)$$

Thereby \mathbf{M} is the finite set $\mathbf{M} = \{1, \dots, n\}$, and $\bar{\mathbf{K}} := \mathbf{M} \setminus \mathbf{K}$ is the complement of $\mathbf{K} \subset \mathbf{M}$. The ON basis systems $\{e_m, m \in \mathbf{M}\} \subset \mathcal{H}_0$ and $\{f_m, m \in \mathbf{M}\} \subset \mathcal{F}_0$ are related by the involution $\mathcal{F}_0 \rightarrow \mathcal{F}_0^*$ and by the linear isometry $J : \mathcal{F}_0^* \rightarrow \mathcal{H}_0$

$$e_m = -J f_m^*, \quad m \in \mathbf{M}. \quad (88)$$

Using the relations (86) and (87) we obtain the identities

$$\begin{aligned}
\Theta a^+(f_m) f_{\mathbf{K}} &= (-1)^{\tau(m, \mathbf{K})} \Theta f_{\mathbf{K} \cup \{m\}} = (-1)^{\tau(m, \mathbf{K}) + \tau(\mathbf{K} + \{m\}, \mathbf{M})} e_{(\bar{\mathbf{K}} \setminus \{m\})} \text{ if } m \in \bar{\mathbf{K}} \\
\Theta a^+(f_m) f_{\mathbf{K}} &= 0 \text{ if } m \in \mathbf{K} \\
a^-(e_m) \Theta f_{\mathbf{K}} &= (-1)^{\tau(\mathbf{K}, \mathbf{M})} a^-(e_m) e_{\bar{\mathbf{K}}} = (-1)^{\tau(\mathbf{K}, \mathbf{M}) + \tau(m, \bar{\mathbf{K}})} e_{(\bar{\mathbf{K}} \setminus \{m\})} \text{ if } m \in \bar{\mathbf{K}} \\
a^-(e_m) \Theta f_{\mathbf{K}} &= (-1)^{\tau(\mathbf{K}, \mathbf{M})} a^-(e_m) e_{\bar{\mathbf{K}}} = 0 \text{ if } m \in \mathbf{K}
\end{aligned}$$

Then the relation $\tau(m, \mathbf{K}) + \tau(\mathbf{K} + \{m\}, \mathbf{M}) = \tau(\mathbf{K}, \mathbf{M}) + \tau(m, \bar{\mathbf{K}}) \bmod 2$ implies $\Theta a^+(f_m) = a^-(e_m) \Theta$ for all $m \in \mathbf{M}$. The mapping (87) has therefore properties

$$\Theta a^\pm(f_m) \Theta^\dagger = a^\mp(e_m), \quad m \in \mathbf{M}. \quad (89)$$

For $h \in \mathcal{F}_0$ a basis expansion $h = \sum_{m \in \mathbf{M}} \gamma_m f_m$, $\gamma_m \in \mathbb{C}$, leads to $a^+(h) = \sum_m \gamma_m a^+(f_m)$ and $a^-(h) = \sum_m \tilde{\gamma}_m a^-(f_m)$. Taking into account the relations (88) the identities (89) get the more abstract form

$$\Theta a^\pm(h) \Theta^\dagger = -a^\mp(J h^*) \text{ if } h \in \mathcal{F}_0. \quad (90)$$

These identities are equivalent to (59).

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